

Zeros and Singularities of Cosmological Wavefunctions



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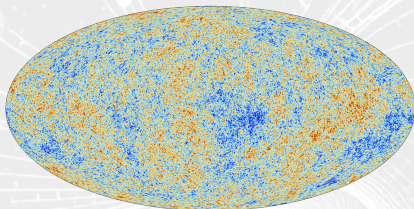
MIAPbP workshop on Primordial Cosmology

Based on [arXiv:2503.23579](#) with S De, A Pokraka, M Spradlin, A Volovich
and [arXiv:2603.08670](#) with M Skowronek, M Spradlin, A Volovich, H-C Weng

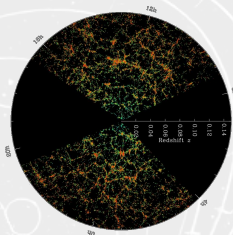
also [work in progress](#) with M Capuano, L Ferro, T Lukowski, A Palazzo, L Ren, M Spradlin, A
Volovich, H-C Weng, Y-Q Zhang

Inflationary Correlators

Important observables in cosmology like the cosmic microwave background and large scale structure



PLANCK



DES

are seeded by **quantum correlations** set up by the end of inflation.

Given a theory of inflation, how do we predict such correlations?

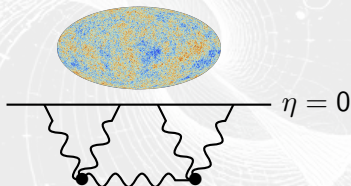
QFT in deSitter

A inflationary cosmology is well-approximated by **de Sitter** spacetime, which is

$$ds^2 = \frac{1}{H^2 \eta^2} (-d\eta^2 + d\mathbf{x}^2)$$

in Poincare patch coordinates.

We are interested in correlators calculated on the **fixed conformal time slice** $\eta = 0$. An example of a contributing Feynman diagram is



Wavefunction of the Universe

Instead of using the interacting picture of QFT, we can instead construct **wavefunctionals**,

$$\Psi[\varphi] = \int_{\phi(-\infty)=0}^{\phi(0)=\varphi} D\phi e^{iS[\phi]}$$

The correlation functions can be derived from it via

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle = \frac{\int D\phi \varphi(x_1) \cdots \varphi(x_n) |\Psi[\varphi]|^2}{\int D\phi |\Psi[\varphi]|^2}$$

Indeed, the object of focus for the rest of this talk will be **wavefunction coefficients** defined as

$$\Psi[\phi] = \exp \left(\sum_{n=2}^{\infty} \int d^3x_1 \cdots d^3x_n \psi_n(x_1, x_2, \cdots, x_n) \varphi(x_1) \varphi(x_2) \cdots \varphi(x_n) \right)$$

and the Feynman graphs that contribute to them.

This Workshop

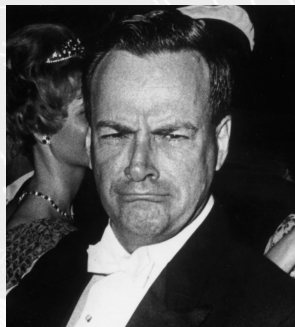
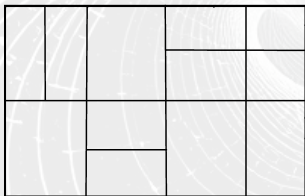
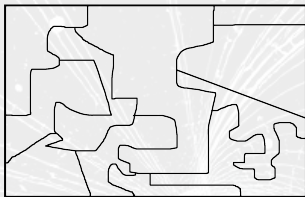
is called “Primordial Cosmology: Novel Perspectives from Scattering Amplitudes, Holography and the Bootstrap”

This Workshop

is called “Primordial Cosmology: Novel Perspectives from **Scattering Amplitudes**, Holography and the Bootstrap”

Feynman Doesn't Rule

The modern amplitudes program has developed a number of techniques that circumvent the calculation of a **combinatorially large** number of Feynman diagrams.



Glimpse at the Calculations by Parke and Taylor (1985)

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As the result of the composition of one hundred and forty Feynman diagrams, we obtain

$$A_{\mathcal{Q}}^T(p_1, p_2, p_3, p_4, p_5) = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \\ p_1 & p_2 & p_3 & p_4 & p_5 \\ p_1 & p_2 & p_3 & p_4 & p_5 \\ p_1 & p_2 & p_3 & p_4 & p_5 \\ p_1 & p_2 & p_3 & p_4 & p_5 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix}, \quad (30)$$

where $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3$, and $\mathbf{U}_4$ are 11-component complex vector functions of the momenta $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$, and $\mathbf{p}_4$; $\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3$, and $\mathbf{U}_4$ are constant 11×11 symmetric matrices. The vectors $\mathbf{U}_1, \mathbf{U}_2$, and $\mathbf{U}_3$ are obtained from the vector $\mathbf{U}$ by the permutations $\{p_1 \leftrightarrow p_2, p_3 \leftrightarrow p_4\}$ and $\{p_1 \leftrightarrow p_3, p_2 \leftrightarrow p_4\}$, respectively, of the momentum variables in $\mathbf{U}$. The individual components of the vector $\mathbf{U}$ represent the sums of all contributions proportional to the appropriately chosen direct color factors. The matrices $\mathbf{K}_i$, which are the suitable sums over the color indices of products of the color factors, contain two independent structures, proportional to $1/3(N^2 - 1)$ and $N(N^2 - 1)$, respectively (N is the number of colors, $N = 3$ for QCD).

Here g denotes the gauge coupling constant. The matrices B^{ab} and K^{ab} are given in table 1. The vector $\mathcal{G}$ is related to the thirty-three diagrams $\mathcal{D}^a$ ($a = 1-33$) for two-gluon-to-four-quark scattering, eleven diagrams $\mathcal{D}^a$ ($a = 1-11$) for two-fermion-to-four-vector scattering and eleven diagrams $\mathcal{D}^a$ ($a = 1-11$) for two-vector-to-four-vector scattering. In the following we:

$$B_1 = \frac{2\lambda_1}{\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6}} \left(\lambda_1^2 C^{(1)} \cdot D_1^2 - \lambda_1\lambda_2 E^{(1)} + \lambda_1\lambda_3 C^{(2)} \cdot D_1^2 \right. \\ \left. - 2\lambda_1 D_1^2 + \lambda_1\lambda_4 \lambda_5 C^{(2)} \cdot D_1^2 \right) \\ B_2 = 2\lambda_1 C^{(1)} \cdot D_1^2 \quad (1)$$

$$E(x, y) = \frac{1}{2} [A_0 x^2 + B_0 xy + C_0 y^2] + \frac{1}{6} [A_1 x^3 + B_1 x^2 y + C_1 xy^2 + D_1 y^3] +$$

$$Df(x, y) = Df(x, p)Df(y, p), \quad (9)$$

J. J. Burke, T. R. Burke / *Journal of Macroeconomics* 25 (2003) 611–628

Wigner $E_{12}^{\text{W}}(E, \mathbf{p}, \mathbf{p}') = 1$											
Matrix E_{12}^{W}						Matrix E_{12}^{W}					
1	2	3	4	5	6	7	8	9	10	11	12
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
7	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
9	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
11	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
12	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

64 J.J. Biele, T.B. Todd / *Journal of Macroeconomics* 24 (2002) 53–64[illegible]

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	

J. Biol. Chem. 276:10101–10107 (2001)

where v is the totally antisymmetric tensor, $v_{\alpha\beta\gamma} = 1$. For the future use, we define one more function,

Fig. 3. *Staphylococcus aureus*—*Enterobacter* spp.

Note that when evaluating A_0 and A_2 at crossed configurations of the momenta, care must be taken with the implicit dependence of the functions E , F and G on the momenta p_1, p_2, p_3, p_4 .

[illegible]

818 *St. Hilaire, T.R. Taylor / Journal of Great Lakes Research 36 (2010) 815–826*

$$\begin{aligned}
 \text{E}(Y_1) &= \frac{4}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^2 \theta \sin^2 \phi \cos^2 \theta \sin^2 \phi \, d\theta \, d\phi \\
 &= \frac{4}{\pi} \int_0^{\pi/2} \sin^2 \phi \, d\phi \int_0^{\pi/2} \cos^4 \theta \, d\theta = \frac{1}{2} \int_0^{\pi/2} \sin^2 \phi \, d\phi \\
 \text{E}(Y_2) &= \frac{4}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^2 \theta \sin^2 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi \\
 &= \frac{4}{\pi} \int_0^{\pi/2} \cos^2 \phi \, d\phi \int_0^{\pi/2} \sin^4 \theta \, d\theta = \frac{1}{2} \int_0^{\pi/2} \cos^2 \phi \, d\phi \\
 \text{E}(Y_1^2) &= \frac{8}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^4 \theta \sin^4 \phi \cos^2 \theta \sin^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_2^2) &= \frac{8}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^4 \theta \sin^4 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_1 Y_2) &= \frac{8}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^4 \theta \sin^4 \phi \cos^2 \theta \sin^2 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_1^3) &= \frac{16}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^6 \theta \sin^6 \phi \cos^2 \theta \sin^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_2^3) &= \frac{16}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^6 \theta \sin^6 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_1^2 Y_2) &= \frac{16}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^6 \theta \sin^6 \phi \cos^2 \theta \sin^2 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_1 Y_2^2) &= \frac{16}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^6 \theta \sin^6 \phi \cos^2 \theta \sin^2 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_1^2 Y_2^2) &= \frac{32}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^8 \theta \sin^8 \phi \cos^2 \theta \sin^2 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_1^3 Y_2) &= \frac{32}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^8 \theta \sin^8 \phi \cos^2 \theta \sin^2 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_1 Y_2^3) &= \frac{32}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^8 \theta \sin^8 \phi \cos^2 \theta \sin^2 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_1^2 Y_2^2 Y_3) &= \frac{64}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^8 \theta \sin^8 \phi \cos^2 \theta \sin^2 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_1^3 Y_2^2) &= \frac{64}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^8 \theta \sin^8 \phi \cos^2 \theta \sin^2 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_1^2 Y_2^3) &= \frac{64}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^8 \theta \sin^8 \phi \cos^2 \theta \sin^2 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi \\
 \text{E}(Y_1^3 Y_2^3) &= \frac{128}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \cos^8 \theta \sin^8 \phi \cos^2 \theta \sin^2 \phi \sin^2 \theta \cos^2 \phi \, d\theta \, d\phi
 \end{aligned}$$

15.1 Study 17.8 Study 1 Case study analysis

[illegible]

Keywords: child sexual abuse; disclosure; self-blame

$$\begin{aligned} \mathcal{D}(1) &= \frac{2}{\pi \sqrt{1-\alpha^2}} \{ F(u_0, \alpha) F(u_0, \alpha) - F(u_0, \alpha) F(u_0, \alpha) \\ &\quad + F(u_0, \alpha) + \alpha_0 F(u_0, \alpha) \}, \\ \mathcal{D}(2) &= \frac{-4}{\pi \sqrt{1-\alpha^2}} \{ F^2(u_0, \alpha) + \alpha_0 F(u_0, \alpha) \\ &\quad + F(u_0, \alpha) + \alpha_0 F(u_0, \alpha) - F(u_0, \alpha) F(u_0, \alpha) \}, \\ \mathcal{D}(3) &= \frac{4}{\pi \sqrt{1-\alpha^2}} \{ F(u_0, \alpha) F(u_0, \alpha) - F(u_0, \alpha) F(u_0, \alpha) \\ &\quad + F(u_0, \alpha) + \alpha_0 F(u_0, \alpha) - F(u_0, \alpha) F(u_0, \alpha) \}. \end{aligned}$$

832 S. J. Davis, T. R. Nadeau / *Journal of Macroeconomics* 25 (2003) 827–842

[illegible]

$$-(\bar{c}_1 + \bar{c}_2 - \bar{c}_3 - \bar{c}_4) \bar{z}^2 (\bar{y} - \bar{z}_1 \bar{z}_2) - (\bar{c}_1 + \bar{c}_2 - \bar{c}_3 - \bar{c}_4) \bar{z}^2 (\bar{y} + \bar{z}_1 \bar{z}_2)$$

$$\begin{aligned} \mathcal{D}(2) &= \frac{1}{\mathcal{N}_2 \mathcal{N}_1 \mathcal{N}_0} [x_2 - x_1 - x_0] x_2 - x_1 + x_0, \\ \mathcal{D}(3) &= \frac{1}{\mathcal{N}_3 \mathcal{N}_2 \mathcal{N}_1} [x_3 - x_2 + x_1 - x_0] x_3 - x_2 + x_1 - x_0, \\ \mathcal{D}(4) &= \frac{1}{\mathcal{N}_4 \mathcal{N}_3 \mathcal{N}_2 \mathcal{N}_1} [x_4 - x_3 + x_2 - x_1 + x_0] x_4 - x_3 + x_2 - x_1 + x_0, \\ \mathcal{D}(5) &= \frac{1}{\mathcal{N}_5 \mathcal{N}_4 \mathcal{N}_3 \mathcal{N}_2 \mathcal{N}_1} [x_5 - x_4 + x_3 - x_2 + x_1 - x_0] x_5 - x_4 + x_3 - x_2 + x_1 - x_0, \\ \mathcal{D}(6) &= \frac{1}{\mathcal{N}_6 \mathcal{N}_5 \mathcal{N}_4 \mathcal{N}_3 \mathcal{N}_2 \mathcal{N}_1} [x_6 - x_5 + x_4 - x_3 + x_2 - x_1 + x_0] x_6 - x_5 + x_4 - x_3 + x_2 - x_1 + x_0, \end{aligned}$$

3.1. Data, 3.2. Models, 3.3. Conclusions and comments

[illegible]

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full area fraction. First the diagrams D are calculated by using eq. (11)–(13). The result is substituted in eq. (6) to obtain the vectors $\mathbf{B}_i$ and $\mathbf{B}_j$. After generating the vectors $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_N, \mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_N$, by the appropriate permutations of moments, eq. (10) is used to obtain the functions A_1 and A_2 . Finally the total area fraction is calculated by using eq. (5). The FORTRAN 2 program based on each of scheme generates one Monte Carlo point in less than a second.

Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multi-loop quantities are tested by obtaining the same quantities from different methods.

Glimpse at the Calculations by Parke and Taylor (1985)

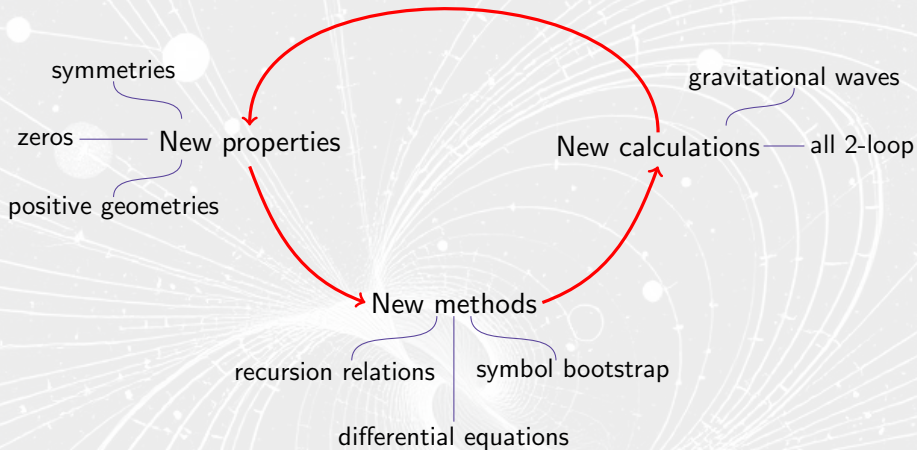
[illegible]

$$\mathcal{A}_n(1^- 2^- 3^+ \dots n^+) = 220 \text{ diagrams} = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n-1, n \rangle \langle n1 \rangle}$$

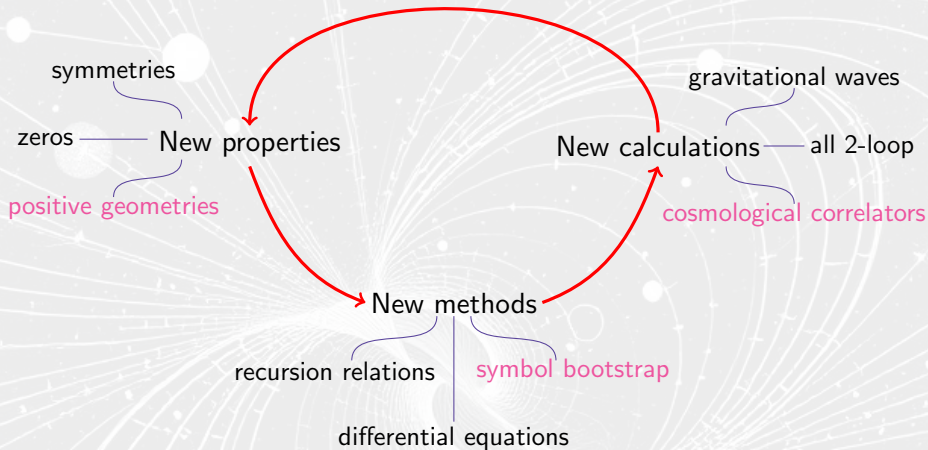
Cyclic nature of studying amplitudes



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An analogy with $\mathcal{N} = 4$ SYM

conformally coupled scalar : cosmological correlators :: $\mathcal{N} = 4$ SYM : amplitudes

- ▶ Tree amplitudes are constructible recursively i.e. Feynman diagrams may be replaced by BCFW terms.

[Britto, Cachazo, Feng, Witten]

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[Arkani-Hamed, Trnka][Damgaard, Ferro, Lukowski, Parisi]

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- ▶ The integrals lead to a space of functions with a well-defined symbol.

[Goncharov, Spradlin, Vergu, Volovich]

- ▶ Symbol bootstrap via properties such as cluster adjacency.

[Caron-Huot, Dixon, Drummond, Dulat, Foster, Gürdoğan, von Hippel, McLeod, Papathanasiou]

⋮

This Talk

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3. What is the space of functions of the integrals?
(Symbols and cluster algebras)

Our model

In this talk, we focus on the following scalar model in de Sitter space:

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{12}R\phi^2 - \sum_p \frac{\lambda_p}{p!} \phi^p \right]$$

Weyl transforming the metric to **flat space** gives:

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So we get the usual flat space Feynman rules except at vertices because the couplings acquire a **scale dependence** as:

$$\lambda_p(\eta) \propto \eta^{(p-4)(1+\epsilon)} \text{ where } \begin{cases} \epsilon = 0 & \text{de Sitter} \\ \epsilon = -1 & \text{flat} \\ \epsilon = -2 & \text{radiation-dominated} \\ \epsilon = -3 & \text{matter-dominated} \end{cases}$$

Our Model

Let us specialize to a **cubic theory** i.e. $p = 3$. This means that the **twist factor** is just ω^ϵ . The coupling can now be rewritten as

$$\lambda_p(\eta) = \int_0^\infty dx \tilde{\lambda}_p(x) e^{i\eta x} = \int_0^\infty dx x^\epsilon e^{i\eta x}$$

Thus at the expense of an **extra integral over x** , we can use flat space Feynman rules:



$$e^{ikt}$$

$$\frac{1}{2k} \left[e^{-ik(t-t')} \theta(t-t') + e^{ik(t-t')} \theta(t'-t) - e^{ik(t+t')} \right]$$

$$\lambda_p(\eta)$$

Flat and deSitter spaces

It is useful to distinguish between partial or internal energies $Y = |\sum \vec{k}_i|$ associated to **edges** in graphs and external ones $X = \sum |\vec{k}_i| = \sum E_i$ associated to **vertices**.

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When the dust settles, we get

$$\psi_{\text{dS}}(X, Y) = \int_0^\infty \prod_v (dx_v x_v^\epsilon) \psi_{\text{flat}}(X_v + x_v, Y_e)$$

where the **integrand is the flat space wavefunction**.

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where the **integrand is the flat space wavefunction**.

We will study the **expansion around de Sitter** (because $\epsilon \approx 0$ is inflation) which is

$$\psi_{\text{dS}}(X, Y) = \int_0^\infty \prod_v dx_v \psi_{\text{flat}}(X_v + x_v, Y_e)$$

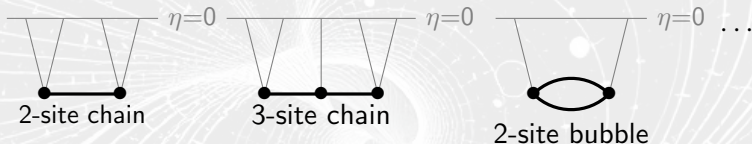
There is a neat combinatorial description of graph contributions via tubings.

Tubings of Graphs

A simple graphical way of calculating the contribution of a diagram is via graph tubings.

$$\psi_{\mathcal{G}} = \sum_{\mathbf{T}} \prod_{\tau \in \mathbf{T}} \frac{1}{S_{\tau}}, \quad \text{where} \quad S_{\tau} = \sum_{\text{sites } v \in \tau} X_v + \sum_{\text{edges } e \text{ that cross } \tau} Y_e,$$

Here $\mathcal{G}$ refers to the **dual graph**:

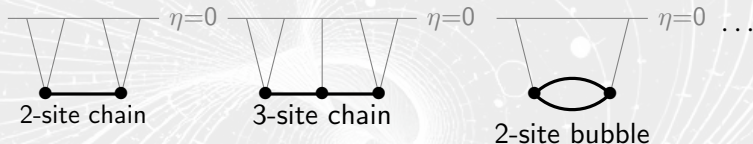


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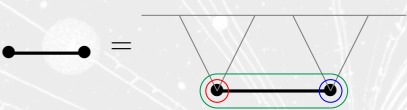


$$\text{wavefunction coefficient} = \sum_{\mathcal{G}} \text{graphs} = \sum_{\mathcal{G}} \sum_{\mathcal{T}} \text{tubings of graphs}$$

[Arkani-Hamed, Benincasa, Postnikov]

Tubings of Graphs: 2-site

A tubing is a set of mutually nested or disjoint tubes that surround a subset of vertices and edges of the graph.



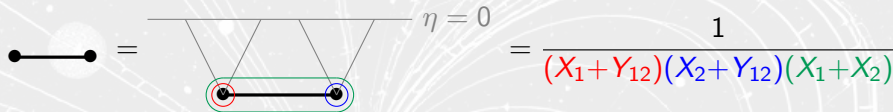
The diagram illustrates a tubing of a graph. On the left, a simple graph with two vertices and one edge is shown. This is equated to a more complex structure. In the center, a graph with two vertices and one edge is shown, where each vertex is enclosed in a colored circle (red and blue). These circles are further enclosed by a larger green rectangle, which is labeled as a tubing. To the right of this diagram is the expression $\eta = 0$. This is equated to a fraction: the numerator is 1, and the denominator is the product of three terms: $(X_1 + Y_{12})$ in red, $(X_2 + Y_{12})$ in blue, and $(X_1 + X_2)$ in green.

$$\text{Graph} = \text{Diagram} \quad \eta = 0 \quad = \frac{1}{(X_1 + Y_{12})(X_2 + Y_{12})(X_1 + X_2)}$$

We saw that poles in the wavefunction occur when partial energies vanish
 $\Rightarrow$ tubes are just partial energies.

Tubings of Graphs: 2-site

A tubing is a set of **mutually nested or disjoint tubes** that surround a **subset of vertices and edges** of the graph.

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We saw that poles in the wavefunction occur when partial energies vanish
 $\Rightarrow$ **tubes are just partial energies.**

Recall that both X and Y are expressible in terms of the external momenta of ϕ as

$$X_1 = |\vec{k}_1| + |\vec{k}_2|$$

$$Y_{12} = |\vec{k}_1 + \vec{k}_2|$$

$$X_2 = |\vec{k}_3| + |\vec{k}_4|$$

$$X_1 + X_2 \neq 0$$

Example: 4-site

Consider a 4-site chain graph. Some of its tubes are common to all tubings:


$$= \frac{1}{\prod_{i=1}^4 (X_i + Y) (\sum_{i=1}^4 X_i)} = \frac{1}{\prod_i S_i S_{\text{tot}}}$$

Example: 4-site

Consider a 4-site chain graph. Some of its tubes are common to all tubings:

$$\text{Diagram: A horizontal line with four black dots. The first and last dots are each enclosed in a blue circle, and the entire line is enclosed in a blue oval.} = \frac{1}{\prod_{i=1}^4 (X_i + Y)(\sum_{i=1}^4 X_i)} = \frac{1}{\prod_i S_i S_{\text{tot}}}$$

These can be grouped together, leaving the “stripped” wavefunction coefficient:

$$\begin{aligned} \text{Diagram: A horizontal line with four black dots. The first two dots are enclosed in a green oval, and the last two dots are enclosed in a pink oval.} &= \frac{1}{(X_1 + X_2 + Y_{23})(X_1 + X_2 + X_3 + Y_{34})} = \frac{1}{S_{12} S_{123}} \\ \text{Diagram: A horizontal line with four black dots. The second and third dots are enclosed in a green oval, and the first and fourth dots are enclosed in a pink oval.} &= \frac{1}{(X_2 + X_3 + Y_{12} + Y_{34})(X_1 + X_2 + X_3 + Y_{34})} = \frac{1}{S_{23} S_{123}} \\ \text{Diagram: A horizontal line with four black dots. The third and fourth dots are enclosed in a green oval, and the first and second dots are enclosed in a pink oval.} &= \frac{1}{(X_2 + X_3 + Y_{12} + Y_{34})(X_2 + X_3 + X_4 + Y_{12})} = \frac{1}{S_{23} S_{234}} \\ \text{Diagram: A horizontal line with four black dots. The third and fourth dots are enclosed in a green oval, and the first and second dots are enclosed in a pink oval.} &= \frac{1}{(X_3 + X_4 + Y_{23})(X_2 + X_3 + X_4 + Y_{12})} = \frac{1}{S_{34} S_{234}} \\ \text{Diagram: A horizontal line with four black dots. The first two dots are enclosed in a pink oval, and the last two dots are enclosed in a pink oval.} &= \frac{1}{(X_1 + X_2 + Y_{23})(X_3 + X_4 + Y_{23})} = \frac{1}{S_{12} S_{34}} \end{aligned}$$

Example:4-site

We see that the 4-site wavefunction is then given by

$$\psi_4 = \frac{1}{s_1 s_2 s_3 s_4 s_{\text{tot}}} \left(\frac{1}{s_{12} s_{123}} + \frac{1}{s_{123} s_{23}} + \frac{1}{s_{234} s_{23}} + \frac{1}{s_{234} s_{34}} + \frac{1}{s_{12} s_{34}} \right)$$

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The usage of S to denote the tubes was intentional because changing S to a 5-point Mandelstam variable s :

$$\xrightarrow{S \rightarrow s} A_5^{\text{Tr}\phi^3} = \frac{1}{s_{123} s_{12}} + \frac{1}{s_{123} s_{23}} + \frac{1}{s_{234} s_{23}} + \frac{1}{s_{234} s_{34}} + \frac{1}{s_{12} s_{34}}$$

gives the planar amplitude of a color-ordered cubic scalar theory, $\text{Tr}\phi^3$.

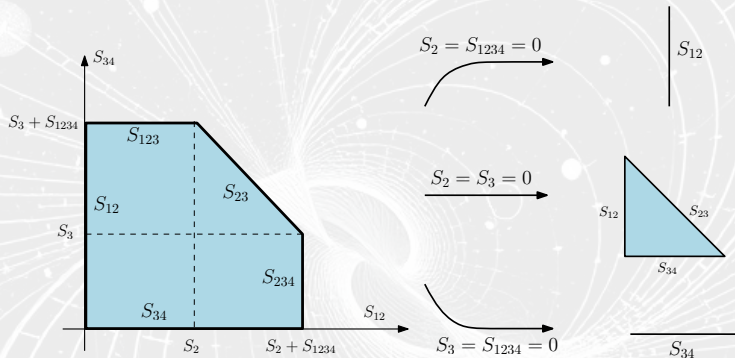
They must be solutions to the same combinatorics problem!

[De, SP, Pokraka, Spradlin, Volovich]

Geometry of the 4-site chain

Using the equivalence, we can associate the 4-site chain to a positive geometry, namely the 5-point $\text{Tr}\phi^3$ associahedron

[Arkani-Hamed, Bai, He, Yan]



[De, SP, Pokraka, Spradlin, Volovich]

What is a geometric description?

- ▶ There is a **positive geometry** (a semi-algebraic subspace) defined wrt the external kinematics.
- ▶ One can define a **canonical form** on this geometry that has **logarithmic singularities** at the boundaries.



The diagram illustrates a sequence of three triangles connected by arrows. The first triangle is a standard right-angled triangle. The second triangle is a right-angled triangle with one side labeled 'a'. The third triangle is a degenerate state where the vertices are marked with dots. Below the triangles, the corresponding mathematical expressions are shown: $\frac{a}{xy(x+y-a)} \xrightarrow{y=0} \frac{a}{x(x-a)} \xrightarrow{x=a} 1.$

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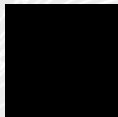
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- ▶ e.g. in physics: amplituhedron, correlahedron, ABJM amplituhedron, EFThedron, associahedron, cosmohedron etc.



A geometry can be associated with a function that gives an integrand

Consequences

- ▶ The canonical form on this $(n-2)$ -dimensional associahedron geometry gives the flat space n -site chain wavefunction coefficient.

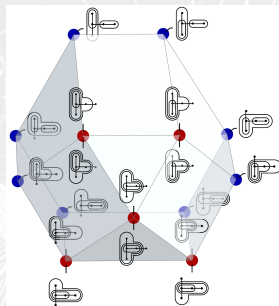
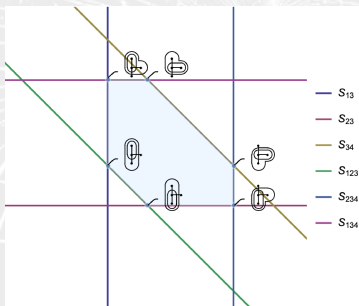
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- ▶ For other graphs, we can build a similar **polytope**, related to faces of the cosmohedron:
[Arkani-Hamed, Figueiredo, Vazao]



(Associated Zeros)

Positive geometries have zeros stemming from their **Minkowski decomposition**.

For cosmological graphs, there are three types

- Wavefunction zeros: These are zeros of chain graphs that can be **inferred from the geometry** e.g. for the 4-site chain:

$$S_{123} + S_{234} = 0$$

$$S_{123} - S_{12} + S_{34} = 0$$

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$$S_1 = S_2 = S_3 = S_4 = 0$$

$$S_{\text{tot}} = 0$$

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$$S_{\text{tot}} = 0$$

- ▶ Factorization zeros: All graphs can be **built recursively out of chains** via factorization and so inherit zeros e.g. for the 4-site chain:

$$S_2 = 0$$

$$S_{123} - S_{12} + S_{34} = 0$$

[De, SP, Pokraka, Spradlin, Volovich], see [Li, Rodina] for extension

Integrals

Recall that for curved space, we have been discussing the **integrand** of the twisted integral:

$$\psi^{\text{dS}} = \int \prod_{i=1}^n (dx_i x_i^\epsilon) \psi_{\text{flat}}(X + x, Y) \approx \int \prod_{i=1}^n dx_i \psi_{\text{flat}}(X + x, Y)$$

For the remainder of the talk, we focus on doing this **integral**.

Solving Integrals: Method of Differential Equations

Consider the **integral family** for the 2-site chain in de Sitter space.

$$\psi_{2,\alpha,\beta,\gamma}^{\text{dS}}(X_1, X_2, Y) = \int dx_1 dx_2 (x_1 x_2)^\epsilon \frac{1}{(X_1 + x_1 + Y)^\alpha (X_2 + x_2 + Y)^\beta (X_1 + X_2)^\gamma}$$

where $\alpha = \beta = \gamma = 1$ for ψ_2^{dS} .

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where $\alpha = \beta = \gamma = 1$ for ψ_2^{dS} .

This integral is a **function of X_1, X_2, Y** , and as a result we can construct a system of differential equations wrt to these parameters containing equations like:

$$\frac{\partial}{\partial X_1} \psi_{2,\alpha,\beta,\gamma}^{\text{dS}} = \alpha \psi_{2,\alpha+1,\beta,\gamma}^{\text{dS}} + \gamma \psi_{2,\alpha,\beta,\gamma+1}^{\text{dS}}$$

Grouping these together smartly gives

$$d\vec{I} = \epsilon A \vec{I}$$

Example: 2-site Chain

For $\epsilon \approx 0$,

$$d\psi_2 = \left[\text{dlog} \left(\frac{X_1 - Y}{X_1 + Y} \right) \right] F_0 + \left[\text{dlog} \left(\frac{X_2 - Y}{X_2 + Y} \right) \right] \tilde{F}_0$$

$$dF_0 = \left[\text{dlog} \left(\frac{X_1 + X_2}{X_2 + Y} \right) \right] Z_0$$

$$d\tilde{F}_0 = \left[\text{dlog} \left(\frac{X_1 + X_2}{X_1 + Y} \right) \right] Z_0$$

$$dZ_0 = 0$$

[Hillman][De, Pokraka][Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel]

We can solve recursively for ψ_2 and we get

$$\psi_2 = -\text{Li}_2 \left(\frac{X_1 - Y}{X_1 + Y} \right) - \text{Li}_2 \left(\frac{X_2 - Y}{X_2 + Y} \right) + \text{Li}_2 \left(\frac{(X_1 - Y)(X_2 - Y)}{(X_1 + Y)(X_2 + Y)} \right) + \frac{\pi^2}{6}$$

[Arkani-Hamed, Maldacena]

Symbols

There is a special class of integrals that appear: iterated integrals also called **Goncharov multiple polylogarithms**

$$G(a_1, a_2, \dots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t)$$

The simplest example is the dilogarithm function

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Differentials of these functions have a special form:

$$dF^{(n)} = \sum_i F_i^{(n-1)} \text{dlog} R_i \Rightarrow \mathcal{S}(F^{(n)}) = \sum_i \mathcal{S}(F_i^{(n-1)}) \otimes R_i$$

The content of this differential is captured by the **symbol** $\mathcal{S}$, e.g.

$$d \text{Li}_2(z) = -\log(1 - z) \text{dlog} z \Rightarrow \mathcal{S}(\text{Li}_2) = -(1 - z) \otimes z.$$

[Goncharov, Spradlin, Vergu, Volovich]

Symbol: 2-site

We can read off the symbol directly from the differential equation:

$$d\psi_2 = \left[\text{dlog} \left(\frac{X_1 - Y}{X_1 + Y} \right) \right] F_0 + \left[\text{dlog} \left(\frac{X_2 - Y}{X_2 + Y} \right) \right] \tilde{F}_0$$

$$dF_0 = \left[\text{dlog} \left(\frac{X_1 + X_2}{X_2 + Y} \right) \right] Z_0$$

$$d\tilde{F}_0 = \left[\text{dlog} \left(\frac{X_1 + X_2}{X_1 + Y} \right) \right] Z_0$$

$$dZ_0 = 0$$

Using the dlog rule, we get:

$$\begin{aligned} \mathcal{S}(\psi_2) &= \mathcal{S}(F_0) \otimes \frac{X_1 - Y}{X_1 + Y} + \mathcal{S}(\tilde{F}_0) \otimes \frac{X_2 - Y}{X_2 + Y} \\ &= \frac{X_1 + X_2}{X_2 + Y} \otimes \frac{X_1 - Y}{X_1 + Y} + \frac{X_1 + X_2}{X_1 + Y} \otimes \frac{X_2 - Y}{X_2 + Y} \end{aligned}$$

which is now the **symbol of the function** we had before.

Symbol Properties

The symbol thus basically encodes the location of branch points and sequences of discontinuities in a function e.g. the first entry is a discontinuity of the function itself.

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Linearity properties of the symbol follow from properties of a logarithm:

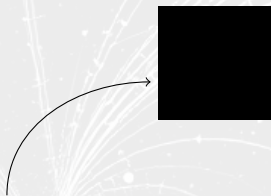
$$(ab) \otimes c = a \otimes c + b \otimes c$$

e.g. since $\log(1) = 0$, we also have

$$a \otimes 1 \otimes b = 0$$

And a given symbol corresponds to a function as long as it satisfies the **“integrability condition”**.

also see [Duhr, Gangl, Rhodes]



A symbol is a sum of sequences of variables separated by $\otimes$ that encode the branch points of an integral

Cosmological Symbols

Kinematic flow gives us a rule to determine what the differential equations are:

$$d\psi = \varepsilon \left[(\psi - F_1) \text{---} \textcircled{\times} \text{---} \text{---} \text{---} + (\psi - \sum F_{2_i}) \text{---} \text{---} \textcircled{\times} \text{---} \text{---} + (\psi - \sum F_{3_i}) \text{---} \text{---} \text{---} \textcircled{\times} \text{---} \right. \\
+ F_1 \text{---} \textcircled{\times} \text{---} \text{---} \text{---} + F_{2_1} \text{---} \textcircled{\times} \text{---} \text{---} \text{---} + F_{3_1} \text{---} \text{---} \textcircled{\times} \text{---} \text{---} \\
(\psi - F_4) \text{---} \text{---} \text{---} \textcircled{\times} \text{---} + F_{2_2} \text{---} \textcircled{\times} \text{---} \text{---} \text{---} + F_{3_2} \text{---} \text{---} \textcircled{\times} \text{---} \text{---} \\
+ F_4 \text{---} \text{---} \textcircled{\times} \text{---} \text{---} + F_{2_3} \text{---} \textcircled{\times} \text{---} \text{---} \text{---} + F_{3_3} \text{---} \text{---} \textcircled{\times} \text{---} \text{---} \Big],$$

It turns out that the symbol letters for a chain graph are then:

$$\sum_{i=j}^k X_i \pm Y_{j-1} \pm Y_k, \quad 1 \leq j \leq k \leq n, \quad \text{and} \quad \sum_{i=1}^n X_i$$

[Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel]

also see [McLeod, Pokraka, Ren]

The only ingredient left is knowing the “Steinmann relations” i.e. which symbol letters can appear next to one another?

Cluster Algebras: A Lightning Introduction

- ▶ Clusters are quivers or **directed graphs with variables at the nodes** of the graphs.

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- ▶ **Mutations modify** a quiver's arrows as well as the variables at each node using a specific mutation rule.
- ▶ The full set of variables generated in this way are called the **cluster variables**.
- ▶ If two variables appear in the same cluster, they are called **cluster compatible**.

[Fomin, Zelevinsky], for review see [Williams]

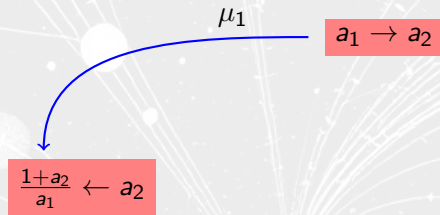
Example of A_2

Start with an initial or **seed cluster** and mutate:

$$a_1 \rightarrow a_2$$

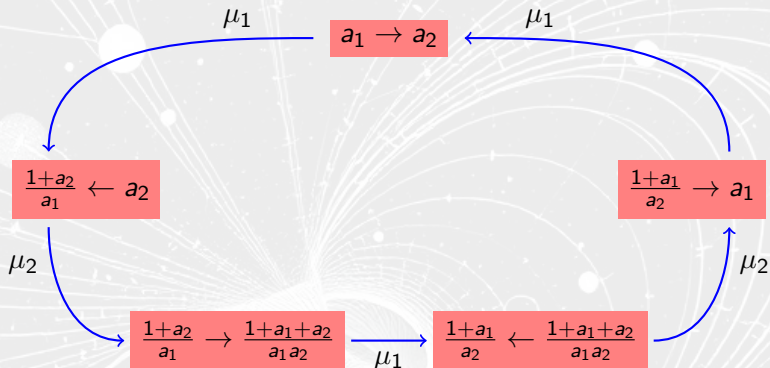
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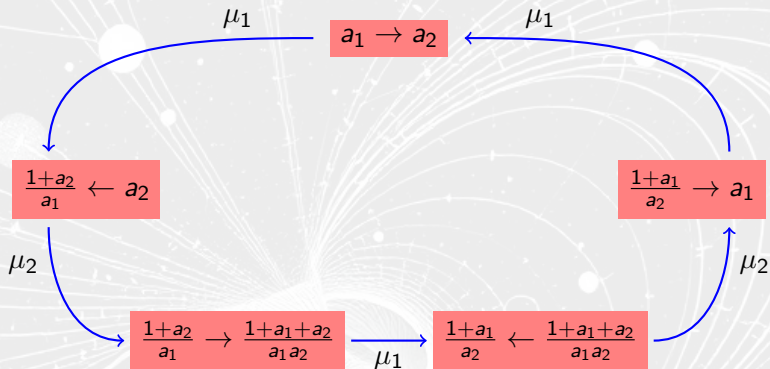
Example of A_2

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Example of A_2

Start with an initial or **seed cluster** and mutate:



Here the cluster variables are:

$$\left\{ a_1, a_2, \frac{1+a_2}{a_1}, \frac{1+a_1+a_2}{a_1 a_2}, \frac{1+a_1}{a_2} \right\} \xrightarrow{\text{change basis}} \{z_1, z_2, 1-z_1, 1-z_2, z_2-z_1\}$$

where consecutive variables are compatible (lie in the same cluster).

Cluster algebras and Amplitudes

How is this related to amplitudes?

- ▶ Cluster variables = symbol alphabet
- ▶ Cluster compatibility = adjacent symbol letters

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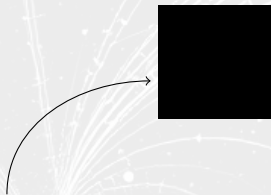
In analogy with $\mathcal{N} = 4$ SYM where

integrand : Grassmannian geometry :: symbol : Grassmannian cluster algebra

[Mago, Schreiber, Spradlin, Volovich][Even-Zohar, Lakrec, Parisi, Tessler, Sherman-Bennet, Williams][Drummond, Foster, Gürdoğan]

was used to bootstrap the symbol up to 8-loops.

[Dixon, Liu][Caron-Huot, Dixon, Dulat, von Hippel, McLeod, Papathanasiou]



Cluster algebras tell us which discontinuities can be taken sequentially i.e.
which variables can appear consecutively in a symbol

2-site symbol alphabet as A_2 cluster variables

Consider the following map from the 2-site symbol alphabet into A_2 :

$$z_1 = \frac{X_1 + Y}{X_1 + X_2}, \quad z_2 = \frac{X_1 - Y}{X_1 + X_2}$$

This reproduces the A_2 cluster algebra:

$$\{1-z_1, z_1, 1-z_2, z_2, z_2-z_1\} = \left\{ \frac{X_2-Y}{X_1+X_2}, \frac{X_1+Y}{X_1+X_2}, \frac{X_2+Y}{X_1+X_2}, \frac{X_1-Y}{X_1+X_2}, \frac{-2Y}{X_1+X_2} \right\}$$

[Mazloumi, Xu][Capuano, Ferro, Lukowski, Palazzo]

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[Mazlouni, Xu][Capuano, Ferro, Lukowski, Palazio]

In addition, the 2-site symbol written as

$$\frac{X_1+Y}{X_1+X_2} \otimes \frac{X_2-Y}{X_1+X_2} + \frac{X_2+Y}{X_1+X_2} \otimes \frac{X_1-Y}{X_1+X_2} - \frac{X_1+Y}{X_1+X_2} \otimes \frac{X_2+Y}{X_1+X_2} - \frac{X_2+Y}{X_1+X_2} \otimes \frac{X_1+Y}{X_1+X_2}$$

is manifestly A_2 -cluster adjacent!

[SP, Skowronek, Spradlin, Volovich, Weng]

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Symbol bootstrap results in a unique answer!

[SP, Skowronek, Spradlin, Volovich, Weng]

n -site Symbol Alphabet

We expect that this continues beyond the 4-site case. We show that the map:

$$z_{2i-1} = \frac{1}{X} \left(\sum_{k=1}^i X_k + Y_i \right), \quad z_{2i} = \frac{1}{X} \left(\sum_{k=1}^i X_k - Y_i \right), \quad 1 \leq i < n$$

maps n -site chain symbol letters into A_{2n-2} cluster variables in such a way that we can use differential equations to prove that:

All chain graphs are A-type cluster adjacent!

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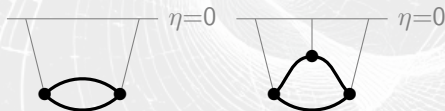
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Similarly, polygonal graphs are B_{2n-1} cluster adjacent!



[SP, Skowronek, Spradlin, Volovich, Weng]

Some bootstrap results: always a unique answer!

For chains:

Constraints	2-site chain		3-site chain		4-site chain	
Integrability and first entry	5	5	102	102	2736	2736
Adjacency	5	N/I	90	N/I	1810	N/I
Flip Symmetry	3	3	47	54	914	1385
$Y_i \rightarrow 0$ limit	1	1	1	1	1	1

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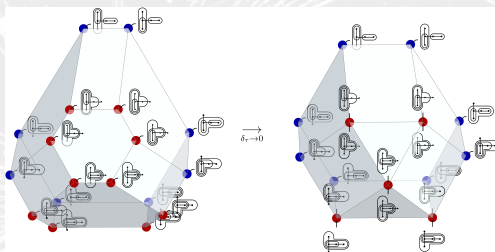
Constraints	2-site loop		3-site loop		4-site loop	
Integrability and first entry	18	18	546	546	23780	23780
Adjacency	17	N/I	379	N/I	9601	N/I
Dihedral Symmetry	7	8	76	111	1267	3137
$Y_i \rightarrow 0$ limit	1	2	1	3	1	7

[SP, Skowronek, Spradlin, Volovich, Weng]

Other graphs

There are three main things to note about other graphs:

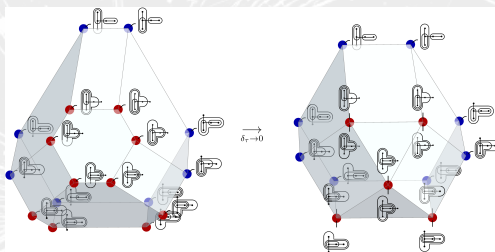
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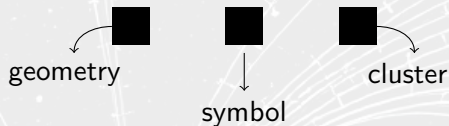
- ▶ The integrand geometries are **no longer simple**. The cosmohedron gives a prescription for a **deformed geometry** that is simple.



- ▶ Kinematic flow tells us the symbol alphabet - but **not** what cluster algebra it could belong to.
- ▶ Understanding unitarity such as via Steinmann relations can help with this.

[Melville][Chowdhury, Jazayeri, Lipstein, Marshall, Mei]

Towards a Full Function



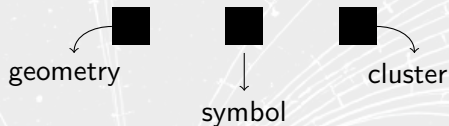
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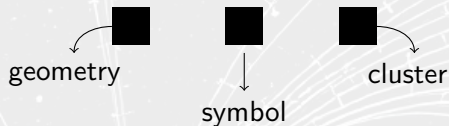
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This can be used to find the weight n result for

All n -site chain wavefunction coefficients!

[Ferro, Lukowski, Ren, Spradlin, Volovich, Weng, Zhang]

Future Directions

WIP [Capuano, Ferro, Lukowski, Palazzo, SP, Ren, Spradlin, Volovich, Weng, Zhang]

- **Higher weight bootstrap:** Going further and further away from the $\epsilon = 0$ de Sitter limit gives higher weight polylogs and hypergeometrics. Can these be bootstrapped too?

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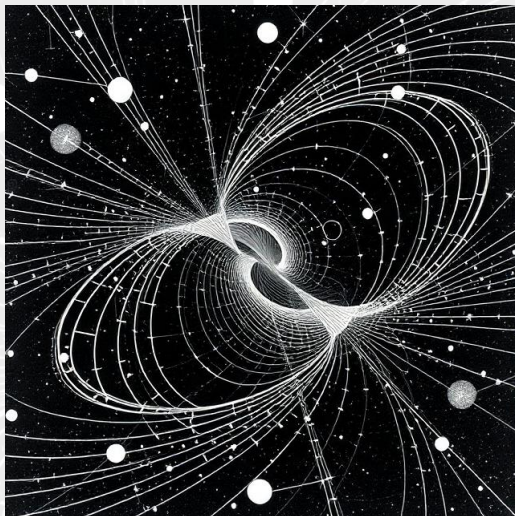
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[Arkani-Hamed, Figueiredo, Vazao][Chowdhury, Lipstein, Marshall, Zhang]

- ▶ **Loops:** Though we have performed some integrals, we have not yet done the integration of loop momenta - the space of functions is actually even richer!

⋮



Thank You