Zeros and Singularities of Cosmological Wavefunctions

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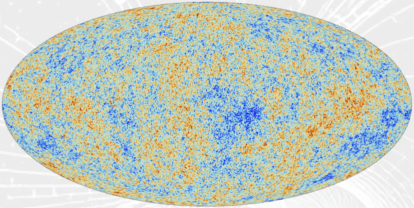
MIAPbP workshop on Primordial Cosmology

Based on arXiv:2503.23579 with S De, A Pokraka, M Spradlin, A Volovich
and arXiv:2603.08670 with M Skowronek, M Spradlin, A Volovich, H-C Weng

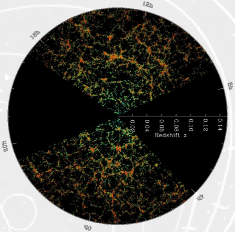
also work in progress with M Capuano, L Ferro, T Lukowski, A Palazio, L Ren, M Spradlin, A Volovich, H-C Weng, Y-Q Zhang

Inflationary Correlators

Important observables in cosmology like the cosmic microwave background and large scale structure



PLANCK



DES

are seeded by quantum correlations set up by the end of inflation.

Given a theory of inflation, how do we predict such correlations?

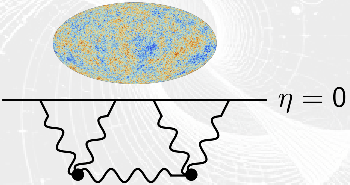
QFT in deSitter

A inflationary cosmology is well-approximated by de Sitter spacetime, which is

$$ds^2 = \frac{1}{H^2\eta^2}(-d\eta^2 + d\mathbf{x}^2)$$

in Poincare patch coordinates.

We are interested in correlators calculated on the fixed conformal time slice $\eta = 0$. An example of a contributing Feynman diagram is



Wavefunction of the Universe

Instead of using the interacting picture of QFT, we can instead construct wavefunctionals,

$$\Psi[\varphi] = \int_{\phi(-\infty)=0}^{\phi(0)=\varphi} D\phi e^{iS[\phi]}$$

The correlation functions can be derived from it via

$$\langle \varphi(x_1)\cdots\varphi(x_n)\rangle = \frac{\int D\phi\varphi(x_1)\cdots\varphi(x_n)|\Psi[\varphi]|^2}{\int D\phi|\Psi[\varphi]|^2}$$

Indeed, the object of focus for the rest of this talk will be wavefunction coefficients defined as

$$\Psi[\phi] = \exp\left(\sum_{n=2}^{\infty}\int d^3x_1\cdots d^3x_n\psi_n(x_1,x_2,\cdots,x_n)\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\right)$$

and the Feynman graphs that contribute to them.

This Workshop

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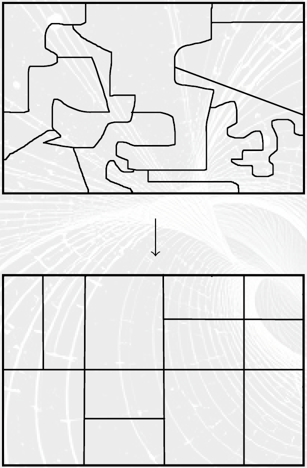
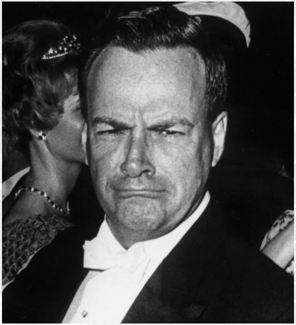
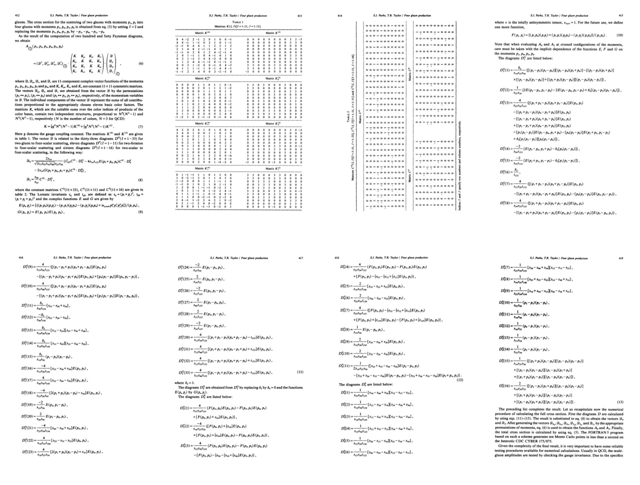
Feynman Doesn't Rule

The modern amplitudes program has developed a number of techniques that circumvent the calculation of a combinatorially large number of Feynman diagrams.





Glimpse at the Calculations by Parke and Taylor (1985)



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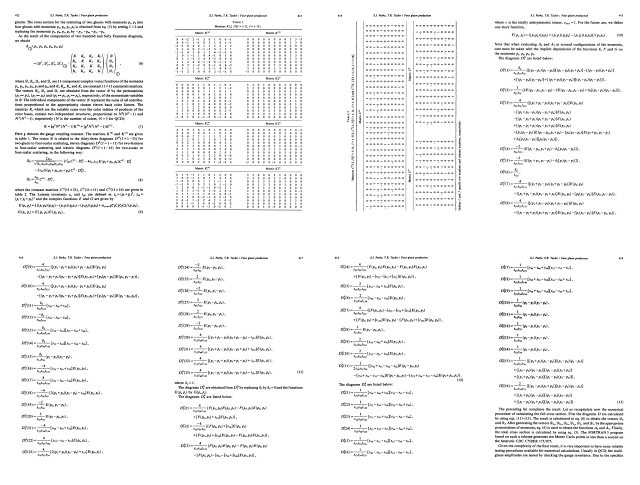
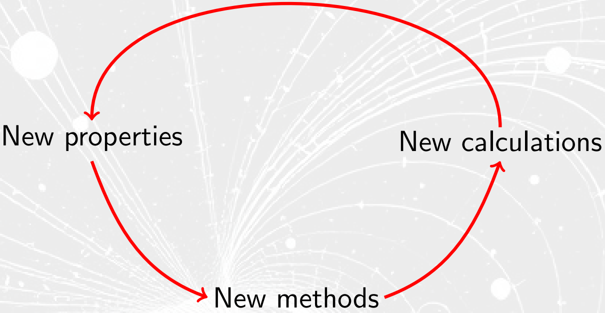


$$\mathcal{A}_n(1^-2^-3^+\cdots n^+) = \text{ 220 diagrams } = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,n\rangle\langle n1\rangle}$$

Cyclic nature of studying amplitudes



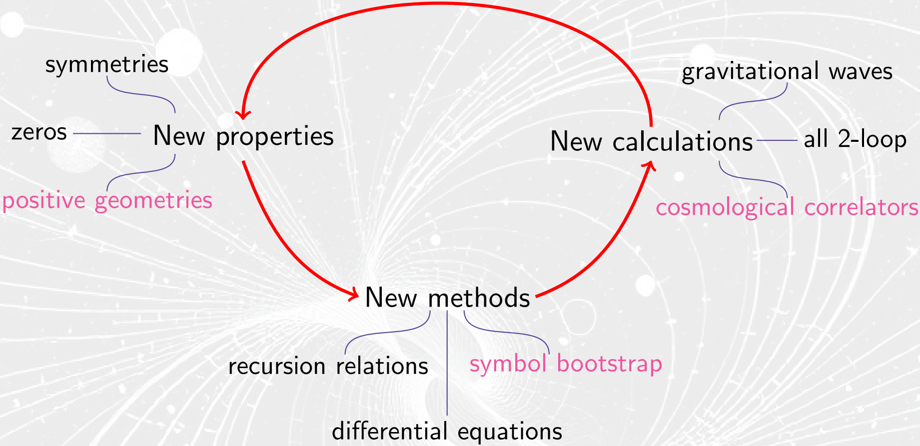
Cyclic nature of studying amplitudes

symmetries
zeros — New properties
positive geometries

New calculations — all 2-loop
gravitational waves

New methods
recursion relations
differential equations
symbol bootstrap

Cyclic nature of studying amplitudes



An analogy with $\mathcal{N} = 4$ SYM

$\boxed{\text{conformally coupled scalar : cosmological correlators :: } \mathcal{N} = 4 \text{ SYM : amplitudes}}$

- Tree amplitudes are constructible recursively i.e. Feynman diagrams may be replaced by BCFW terms.

[Britto, Cachazo, Feng, Witten]

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- The integrals lead to a space of functions with a well-defined symbol.
  [Goncharov, Spradlin, Vergu, Volovich]
- Symbol bootstrap via properties such as cluster adjacency.
  [Caron-Huot, Dixon, Drummond, Dulat, Foster, Gürdoğan, von Hippel, McLeod, Papathanasiou]

$\vdots$

This Talk

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2. Is there a geometric description of the integrands? (Amplituhedron)
3. What is the space of functions of the integrals? (Symbols and cluster algebras)

Our model

In this talk, we focus on the following scalar model in de Sitter space:

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{12}R\phi^2 - \sum_p \frac{\lambda_p}{p!}\phi^p \right]$$

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So we get the usual flat space Feynman rules except at vertices because the couplings acquire a scale dependence as:

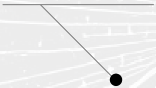
$$\lambda_p(\eta) \propto \eta^{(p-4)(1+\epsilon)} \text{ where } \begin{cases} \epsilon = 0 & \text{de Sitter} \\ \epsilon = -1 & \text{flat} \\ \epsilon = -2 & \text{radiation-dominated} \\ \epsilon = -3 & \text{matter-dominated} \end{cases}$$

Our Model

Let us specialize to a cubic theory i.e. $p = 3$. This means that the twist factor is just ω^{ϵ}. The coupling can now be rewritten as

$$\lambda_p(\eta) = \int_0^{\infty} dx \tilde{\lambda}_p(x) e^{i\eta x} = \int_0^{\infty} dx \; x^{\epsilon} e^{i\eta x}$$

Thus at the expense of an extra integral over x, we can use flat space Feynman rules:



e^{ikt}

$$\frac{1}{2k}\left[e^{-ik(t-t')}\theta(t-t') + e^{ik(t-t')}\theta(t'-t) - e^{ik(t+t')}\right]$$

$\lambda_p(\eta)$

Flat and deSitter spaces

It is useful to distinguish between partial or internal energies $Y = |\sum \vec{k}_i|$ associated to edges in graphs and external ones $X = \sum |\vec{k}_i| = \sum E_i$ associated to vertices.

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When the dust settles, we get

$$\psi_{\text{dS}}(X, Y) = \int_0^\infty \prod_v (dx_v x_v^\epsilon) \ \psi_{\text{flat}}(X_v + x_v, Y_e)$$

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We will study the expansion around de Sitter (because $\epsilon \approx 0$ is inflation) which is

$$\psi_{\mathrm{dS}}(X, Y) = \int_0^\infty \prod_v dx_v \ \psi_{\mathrm{flat}}(X_v + x_v, Y_e)$$

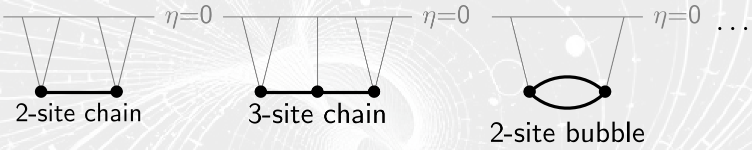
There is a neat combinatorial description of graph contributions via tubings.

Tubings of Graphs

A simple graphical way of calculating the contribution of a diagram is via graph tubings.

$$\psi_{\mathcal{G}} = \sum_{\mathsf{T}} \prod_{\tau \in \mathsf{T}} \frac{1}{S_\tau}, \qquad \text{where} \qquad S_\tau = \sum_{\text{sites } v \in \tau} X_v + \sum_{\text{edges } e \text{ that cross } \tau} Y_e\,,$$

Here $\mathcal{G}$ refers to the dual graph:



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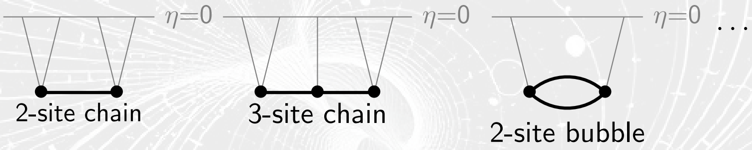


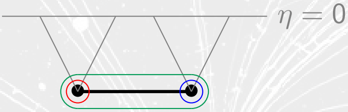
$$\text{wavefunction coefficient} = \sum_{\mathcal{G}} \text{graphs} = \sum_{\mathcal{G}} \sum_{T} \text{tubings of graphs}$$

[Arkani-Hamed, Benincasa, Postnikov]

Tubings of Graphs: 2-site

A tubing is a set of mutually nested or disjoint tubes that surround a subset of vertices and edges of the graph.

$=$



$$= \frac{1}{(X_1+Y_{12})(X_2+Y_{12})(X_1+X_2)}$$

We saw that poles in the wavefunction occur when partial energies vanish $\Rightarrow$ tubes are just partial energies.

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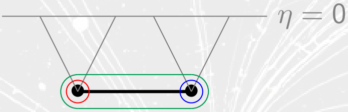

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Recall that both X and Y are expressible in terms of the external momenta of ϕ as

$$X_1 = |\vec{k}_1| + |\vec{k}_2| \qquad X_2 = |\vec{k}_3| + |\vec{k}_4|$$

$$Y_{12} = |\vec{k}_1 + \vec{k}_2| \qquad X_1 + X_2 \neq 0$$

Example: 4-site

Consider a 4-site chain graph. Some of its tubes are common to all tubings:

$$= \frac{1}{\prod_{i=1}^{4}(X_i + Y)(\sum_{i=1}^{4} X_i)} = \frac{1}{\prod_i S_i S_{\text{tot}}}$$

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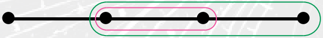
These can be grouped together, leaving the "stripped" wavefunction coefficient:

$$= \frac{1}{(X_1+X_2+Y_{23})(X_1+X_2+X_3+Y_{34})} = \frac{1}{S_{12}S_{123}}$$

$$= \frac{1}{(X_2+X_3+Y_{12}+Y_{34})(X_1+X_2+X_3+Y_{34})} = \frac{1}{S_{23}S_{123}}$$



$$= \frac{1}{(X_2+X_3+Y_{12}+Y_{34})(X_2+X_3+X_4+Y_{12})} = \frac{1}{S_{23}S_{234}}$$

$$= \frac{1}{(X_3+X_4+Y_{23})(X_2+X_3+X_4+Y_{12})} = \frac{1}{S_{34}S_{234}}$$

$$= \frac{1}{(X_1+X_2+Y_{23})(X_3+X_4+Y_{23})} = \frac{1}{S_{12}S_{34}}$$

Example:4-site

We see that the 4-site wavefunction is then given by

$$\psi_4 = \frac{1}{S_1 S_2 S_3 S_4 S_{\text{tot}}} \left(\frac{1}{S_{12} S_{123}} + \frac{1}{S_{123} S_{23}} + \frac{1}{S_{234} S_{23}} + \frac{1}{S_{234} S_{34}} + \frac{1}{S_{12} S_{34}} \right)$$

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The usage of S to denote the tubes was intentional because changing S to a 5-point Mandelstam variable s:

$$\xrightarrow{S \to s} \qquad A_5^{\text{Tr}\phi^3} = \frac{1}{s_{123} s_{12}} + \frac{1}{s_{123} s_{23}} + \frac{1}{s_{234} s_{23}} + \frac{1}{s_{234} s_{34}} + \frac{1}{s_{12} s_{34}}$$

gives the planar amplitude of a color-ordered cubic scalar theory, $\text{Tr}\phi^3$.

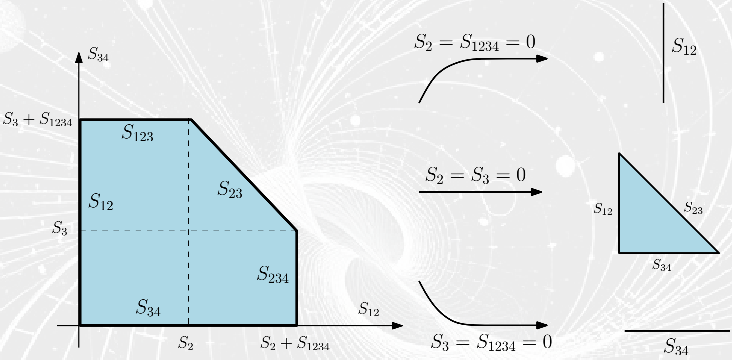
They must be solutions to the same combinatorics problem!

[De, SP, Pokraka, Spradlin, Volovich]

Geometry of the 4-site chain

Using the equivalence, we can associate the 4-site chain to a positive geometry, namely the 5-point Trϕ^3 associahedron

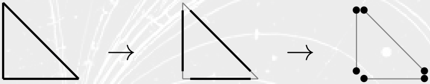
[Arkani-Hamed, Bai, He, Yan]



[De, SP, Pokraka, Spradlin, Volovich]

What is a geometric description?

- There is a positive geometry (a semi-algebraic subspace) defined wrt the external kinematics.
- One can define a canonical form on this geometry that has logarithmic singularities at the boundaries.



$$\frac{a}{xy(x+y-a)} \overset{y=0}{\rightarrow} \frac{a}{x(x-a)} \overset{x=a}{\rightarrow} 1.$$

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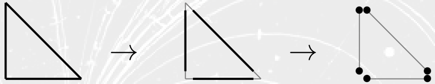

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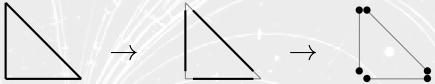

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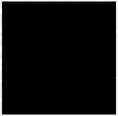
- e.g. in physics: amplituhedron, correlahedron, ABJM amplituhedron, EFThedron, associahedron, cosmohedron etc.



A geometry can be associated with a function that gives an integrand

Consequences

- The canonical form on this $(n-2)$-dimensional associahedron geometry gives the flat space n-site chain wavefunction coefficient.

$$\psi_n = \frac{1}{S_{\text{tot}} \prod_{i=1}^{n} S_i} \sum_{\text{vertices}} \frac{1}{\prod_{\text{edges } e} S_e}$$

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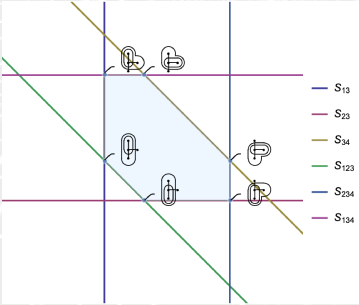
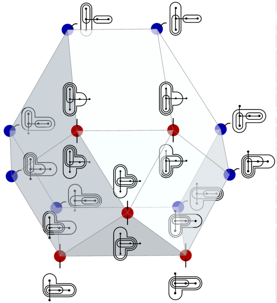
- For other graphs, we can build a similar polytope, related to faces of the cosmohedron: [Arkani-Hamed, Figueiredo, Vazao]





(Associated Zeros)

Positive geometries have zeros stemming from their Minkowski decomposition.

For cosmological graphs, there are three types

- Wavefunction zeros: These are zeros of chain graphs that can be inferred from the geometry e.g. for the 4-site chain:

$$S_{123} + S_{234} = 0 \qquad\qquad S_{123} - S_{12} + S_{34} = 0$$

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$$S_1 = S_2 = S_3 = S_4 = 0 \qquad\qquad S_{\text{tot}} = 0$$

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$$S_1 = S_2 = S_3 = S_4 = 0 \qquad\qquad S_{\text{tot}} = 0$$

- Factorization zeros: All graphs can be built recursively out of chains via factorization and so inherit zeros e.g. for the 4-site chain:

$$S_2 = 0 \qquad\qquad S_{123} - S_{12} + S_{34} = 0$$

[De, SP, Pokraka, Spradlin, Volovich], see [Li, Rodina] for extension

Integrals

Recall that for curved space, we have been discussing the integrand of the twisted integral:

$$\psi^{\mathrm{dS}} = \int \prod_{i=1}^{n} (dx_i x_i^{\epsilon}) \psi_{\mathrm{flat}}(X + x, Y) \approx \int \prod_{i=1}^{n} dx_i \; \psi_{\mathrm{flat}}(X + x, Y)$$

For the remainder of the talk, we focus on doing this integral.

Solving Integrals: Method of Differential Equations

Consider the integral family for the 2-site chain in de Sitter space.

$$\psi^{\mathrm{dS}}_{2,\alpha,\beta,\gamma}(X_1, X_2, Y) = \int dx_1 dx_2 (x_1 x_2)^{\epsilon} \frac{1}{(X_1 + x_1 + Y)^{\alpha}(X_2 + x_2 + Y)^{\beta}(X_1 + X_2)^{\gamma}}$$

where $\alpha = \beta = \gamma = 1$ for ψ^{dS}_2.

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where $\alpha = \beta = \gamma = 1$ for ψ^{dS}_2.

This integral is a function of X_1, X_2, Y, and as a result we can construct a system of differential equations wrt to these parameters containing equations like:

$$\frac{\partial}{\partial X_1} \psi^{\mathrm{dS}}_{2,\alpha,\beta,\gamma} = \alpha \psi^{\mathrm{dS}}_{2,\alpha+1,\beta,\gamma} + \gamma \psi^{\mathrm{dS}}_{2,\alpha,\beta,\gamma+1}$$

Grouping these together smartly gives

$$\mathrm{d}\vec{I} = \epsilon A \vec{I}$$

Example: 2-site Chain

For $\epsilon \approx 0$,

$$\begin{aligned}
\mathrm{d}\psi_2 &= \left[\mathrm{dlog}\left(\frac{X_1 - Y}{X_1 + Y}\right)\right] F_0 + \left[\mathrm{dlog}\left(\frac{X_2 - Y}{X_2 + Y}\right)\right] \tilde{F}_0 \\
\mathrm{d}F_0 &= \left[\mathrm{dlog}\left(\frac{X_1 + X_2}{X_2 + Y}\right)\right] Z_0 \\
\mathrm{d}\tilde{F}_0 &= \left[\mathrm{dlog}\left(\frac{X_1 + X_2}{X_1 + Y}\right)\right] Z_0 \\
dZ_0 &= 0
\end{aligned}$$

[Hillman][De, Pokraka][Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel]

We can solve recursively for ψ_2 and we get

$$\psi_2 = -\mathrm{Li}_2\left(\frac{X_1 - Y}{X_1 + Y}\right) - \mathrm{Li}_2\left(\frac{X_2 - Y}{X_2 + Y}\right) + \mathrm{Li}_2\left(\frac{(X_1 - Y)(X_2 - Y)}{(X_1 + Y)(X_2 + Y)}\right) + \frac{\pi^2}{6}$$

[Arkani-Hamed, Maldacena]

Symbols

There is a special class of integrals that appear: iterated integrals also called Goncharov multiple polylogarithms

$$G(a_1, a_2, \cdots a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \cdots, a_n; t)$$

The simplest example is the dilogarithm function

$$\mathrm{Li}_2(z) = -\int_0^z \frac{dt}{t} \log(1 - t)$$

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Differentials of these functions have a special form:

$$\mathrm{d}F^{(n)} = \sum_i F_i^{(n-1)} \mathrm{dlog} R_i \Rightarrow \mathcal{S}(F^{(n)}) = \sum_i \mathcal{S}(F_i^{(n-1)}) \otimes R_i$$

The content of this differential is captured by the symbol $\mathcal{S}$, e.g.

$$\mathrm{d}\,\mathrm{Li}_2(z) = -\log(1-z)\,\mathrm{dlog}\,z \Rightarrow \mathcal{S}(\mathrm{Li}_2) = -(1-z) \otimes z.$$

[Goncharov, Spradlin, Vergu, Volovich]

Symbol: 2-site

We can read off the symbol directly from the differential equation:

$$
\begin{aligned}
\mathrm{d}\psi_2 &= \left[\mathrm{dlog}\left(\frac{X_1 - Y}{X_1 + Y}\right)\right] F_0 + \left[\mathrm{dlog}\left(\frac{X_2 - Y}{X_2 + Y}\right)\right] \tilde{F}_0 \\
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dZ_0 &= 0
\end{aligned}
$$

Using the dlog rule, we get:

$$
\begin{aligned}
\mathcal{S}(\psi_2) =& \mathcal{S}(F_0) \otimes \frac{X_1 - Y}{X_1 + Y} + \mathcal{S}(\tilde{F}_0) \otimes \frac{X_2 - Y}{X_2 + Y} \\
=& \frac{X_1 + X_2}{X_2 + Y} \otimes \frac{X_1 - Y}{X_1 + Y} + \frac{X_1 + X_2}{X_1 + Y} \otimes \frac{X_2 - Y}{X_2 + Y}
\end{aligned}
$$

which is now the symbol of the function we had before.

Symbol Properties

The symbol thus basically encodes the location of branch points and sequences of discontinuities in a function e.g. the first entry is a discontinuity of the function itself.

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Linearity properties of the symbol follow from properties of a logarithm:

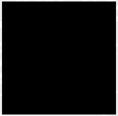
$$(ab) \otimes c = a \otimes c + b \otimes c$$

e.g. since $\log(1) = 0$, we also have

$$a \otimes 1 \otimes b = 0$$

And a given symbol corresponds to a function as long as it satisfies the "integrability condition".

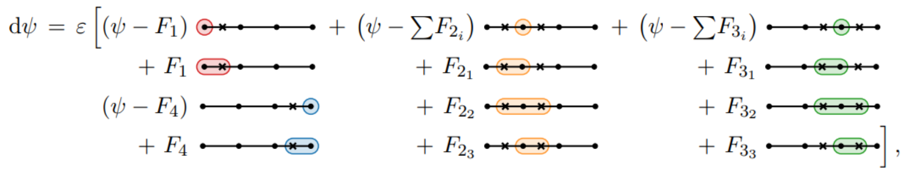
also see [Duhr, Gangl, Rhodes]



A symbol is a sum of sequences of variables separated by $\otimes$ that encode the branch points of an integral

Cosmological Symbols

Kinematic flow gives us a rule to determine what the differential equations are:



It turns out that the symbol letters for a chain graph are then:

$$\sum_{i=j}^{k} X_i \pm Y_{j-1} \pm Y_k \,, \quad 1 \leq j \leq k \leq n \,, \quad \text{and} \quad \sum_{i=1}^{n} X_i$$

[Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel]

also see [McLeod, Pokraka, Ren]

The only ingredient left is knowing the "Steinmann relations" i.e. which symbol letters can appear next to one another?

Cluster Algebras: A Lightning Introduction

- Clusters are quivers or directed graphs with variables at the nodes of the graphs.

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- Mutations modify a quiver's arrows as well as the variables at each node using a specific mutation rule.
- The full set of variables generated in this way are called the cluster variables.
- If two variables appear in the same cluster, they are called cluster compatible.

[Fomin, Zelevinsky], for review see [Williams]

Example of A_2

Start with an initial or seed cluster and mutate:

$$\boxed{a_1 \longrightarrow a_2}$$

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& & \boxed{a_1 \to a_2} \arrow[dll, bend right, "\mu_1"'] & & \\
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\end{tikzcd}
$$

Here the cluster variables are:

$$\left\{ a_1, a_2, \frac{1+a_2}{a_1}, \frac{1+a_1+a_2}{a_1 a_2}, \frac{1+a_1}{a_2} \right\} \xrightarrow{\text{change basis}} \{z_1, z_2, 1-z_1, 1-z_2, z_2-z_1\}$$

where consecutive variables are compatible (lie in the same cluster).

Cluster algebras and Amplitudes

How is this related to amplitudes?

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Cluster algebras and Amplitudes

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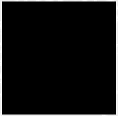
In analogy with $\mathcal{N} = 4$ SYM where

$$\boxed{\text{integrand} : \text{Grassmannian geometry} :: \text{symbol} : \text{Grassmannian cluster algebra}}$$

[Mago, Schreiber, Spradlin, Volovich][Even-Zohar, Lakrec, Parisi, Tessler, Sherman-Bennet, Williams][Drummond, Foster, Gürdoǧan]

was used to bootstrap the symbol up to 8-loops.

[Dixon, Liu][Caron-Huot, Dixon, Dulat, von Hippel, McLeod, Papathanasiou]



Cluster algebras tell us which discontinuities can be taken sequentially i.e. which variables can appear consecutively in a symbol

2-site symbol alphabet as A_2 cluster variables

Consider the following map from the 2-site symbol alphabet into A_2:

$$z_1 = \frac{X_1 + Y}{X_1 + X_2}, \quad z_2 = \frac{X_1 - Y}{X_1 + X_2}$$

This reproduces the A_2 cluster algebra:

$$\{1-z_1,\, z_1,\, 1-z_2,\, z_2,\, z_2-z_1\} = \left\{\frac{X_2-Y}{X_1+X_2}, \frac{X_1+Y}{X_1+X_2}, \frac{X_2+Y}{X_1+X_2}, \frac{X_1-Y}{X_1+X_2}, \frac{-2Y}{X_1+X_2}\right\}$$

[Mazloumi, Xu][Capuano, Ferro, Lukowski, Palazio]

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[Mazloumi, Xu][Capuano, Ferro, Lukowski, Palazio]

In addition, the 2-site symbol written as

$$\frac{X_1+Y}{X_1+X_2} \otimes \frac{X_2-Y}{X_1+X_2} + \frac{X_2+Y}{X_1+X_2} \otimes \frac{X_1-Y}{X_1+X_2} - \frac{X_1+Y}{X_1+X_2} \otimes \frac{X_2+Y}{X_1+X_2} - \frac{X_2+Y}{X_1+X_2} \otimes \frac{X_1+Y}{X_1+X_2}$$

is manifestly A_2-cluster adjacent!

[SP, Skowronek, Spradlin, Volovich, Weng]

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- Start with the most general possible weight 2 symbol that can be built out of our letters $\supset \ z_1 \otimes (1 - z_1)$: 25 possible.

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Symbol bootstrap results in a unique answer!

[SP, Skowronek, Spradlin, Volovich, Weng]

n-site Symbol Alphabet

We expect that this continues beyond the 4-site case. We show that the map:

$$z_{2i-1} = \frac{1}{X}\left(\sum_{k=1}^{i} X_k + Y_i\right), \quad z_{2i} = \frac{1}{X}\left(\sum_{k=1}^{i} X_k - Y_i\right), \qquad 1 \leq i < n$$

maps n-site chain symbol letters into A_{2n-2} cluster variables in such a way that we can use differential equations to prove that:

All chain graphs are A-type cluster adjacent!

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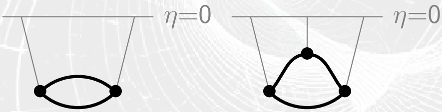
Similarly, polygonal graphs are B_{2n-1} cluster adjacent!



[SP, Skowronek, Spradlin, Volovich, Weng]

Some bootstrap results: always a unique answer!

For chains:

Constraints	2-site chain		3-site chain		4-site chain	
Integrability and first entry	5	5	102	102	2736	2736
Adjacency	5	N/I	90	N/I	1810	N/I
Flip Symmetry	3	3	47	54	914	1385
$Y_i \to 0$ limit	1	1	1	1	1	1

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For polygons:

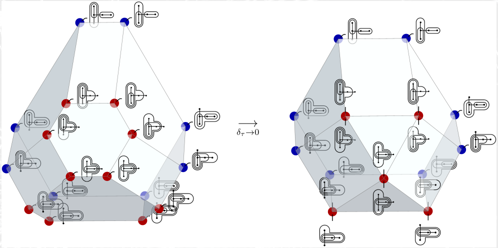
Constraints	2-site loop		3-site loop		4-site loop	
Integrability and first entry	18	18	546	546	23780	23780
Adjacency	17	N/I	379	N/I	9601	N/I
Dihedral Symmetry	7	8	76	111	1267	3137
$Y_i \to 0$ limit	1	2	1	3	1	7

[SP, Skowronek, Spradlin, Volovich, Weng]

Other graphs

There are three main things to note about other graphs:

- The integrand geometries are no longer simple. The cosmohedron gives a prescription for a deformed geometry that is simple.



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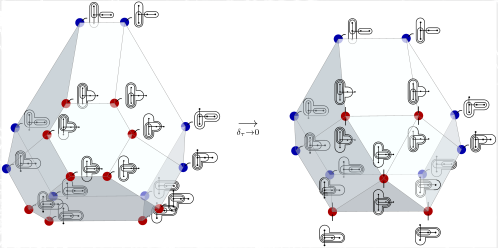
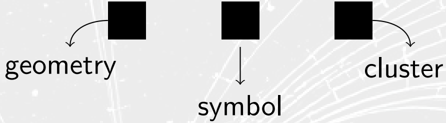


- Kinematic flow tells us the symbol alphabet - but not what cluster algebra it could belong to.
- Understanding unitarity such as via Steinmann relations can help with this.

[Melville][Chowdhury, Jazayeri, Lipstein, Marshall, Mei]

Towards a Full Function



To get from the symbol to the function, we note that cosmological symbols actually have global (not just adjacent) cluster compatibility.

$$a_1 \otimes \cdots \underbrace{a_i \otimes \cdots a_j}_{\text{compatible}} \otimes a_n \text{ even if } j \neq i+1$$

[Capuano, Ferro, Lukowski, Palazio, Zhang]

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Globally A-type cluster compatible functions have been classified.

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This can be used to find the weight n result for

All n-site chain wavefunction coefficients!

[Ferro, Lukowski, Ren, Spradlin, Volovich, Weng, Zhang]

Future Directions

WIP [Capuano, Ferro, Lukowski, Palazio, SP, Ren, Spradlin, Volovich, Weng, Zhang]

- Higher weight bootstrap: Going further and further away from the $\epsilon = 0$ de Sitter limit gives higher weight polylogs and hypergeometrics. Can these be bootstrapped too?

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[Baumann, Joyce, Lee, Vaziri]

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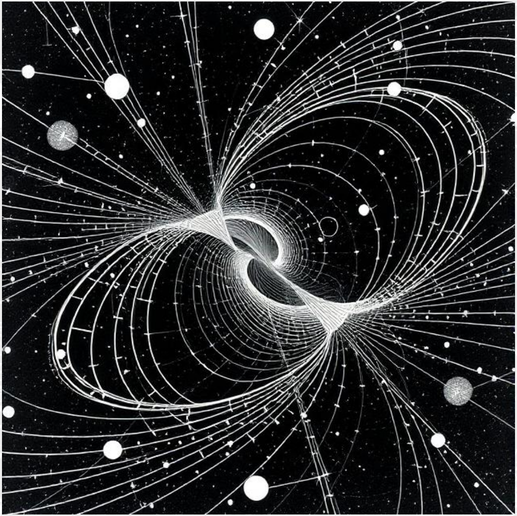
  [Arkani-Hamed, Figueiredo, Vazao][Chowdhury, Lipstein, Marshall, Zhang]
- Loops: Though we have performed some integrals, we have not yet done the integration of loop momenta - the space of functions is actually even richer!

$$\vdots$$



Thank You