Infinities and Zeros of Scalar Amplitudes

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Based on arXiv:2504.11253 with M Skowronek, M Spradlin, A Volovich
and arXiv:2505.02520 with C Jones

Map of the Talk

- Introduce special directions in kinematic space called g-vectors
- Fully characterize the behavior of Tr ϕ^3 amplitudes at infinity
- Discuss the properties of this function at infinity: is it an amplitude?
- Understand the related deformation of the underlying positive geometry (associahedron)
- Construct recursion relations that manifest hidden zeros and near-zero splitting in a number of theories

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[Britto, Cachazo, Feng, Witten, Benincases, Boucher-Veronneau, Arkani-Hamed, Kaplan, Bern, Davies, Dennan, Huang, Herrmann, Trnka, Brown, Oktem...]

Understanding Amplitudes at Infinity

Poles at finite locations are well-understood.

- Locality $\Rightarrow$ poles when intermediate momenta go on-shell.
- Unitarity $\Rightarrow$ amplitude factorizes on these poles.

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? = Lack of pole at infinity

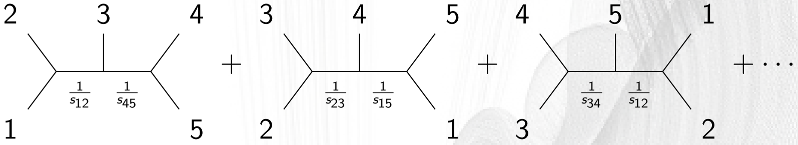
Like BCFW: Lake of pole at infinity $\Rightarrow$ recursion relations

Good Directions and Feynman Diagrams

The main theory of interest is Tr ϕ^3

$$\mathcal{L} = Tr(\phi_{ab}\phi_{bc}\phi_{ca})$$

The 5-point amplitude has many Feynman diagrams



For the amplitude to have no pole at infinity, we must take the propagators to infinity in a direction such that

$$\mathcal{A}_n \xrightarrow[z\to\infty]{s_i \to s_i + a_i z} \mathcal{O}\left(\frac{1}{z^2}\right)$$

Kinematic Variables in Planar Scattering

Color-ordered amplitudes lend themselves to a nice kinematic variables:

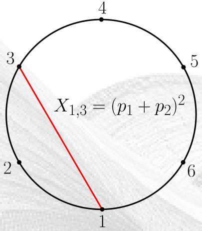
$$X_{ij} = (p_i + \cdots p_{j-1})^2$$

Massless $\Rightarrow X_{ii+1} = 0$ and momentum conservation $\Rightarrow X_{ij} = X_{ji}$. These are independent in high enough dimension.

There are also non-planar variables $c_{ij} = (p_i + p_j)^2$ which satisfy:

$$c_{ij} = X_{ij} + X_{i+1j+1} - X_{ij+1} - X_{i+1j}\,.$$

Planarity allows us to organize X's on a surface:



- Every triangulation is a valid Feynman diagram.
- Factorization splits the surface in two.

g-vector Shifts

Choose a basis of kinematic variables containing

- $X_{ij} \in \mathcal{T}$: $n-3$ chords that form a triangulation.
- $c_{ij} \in \mathcal{T}$ if $X_{i+1j+1} \notin \mathcal{T}$: All but $n-3$ non-planar variables complete the basis.

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We now consider a shift $(\mathcal{T}, \vec{t})$ where

- $X_{ij} \to X_{ij} + \vec{t}_{ij} z$ for all X in the basis.
- $c_{ij} \in$ basis do not shift.
- X_{ij} not in basis shift such that the c_{ij} can remain unshifted.

This is an alternate way of saying $X \to X + \vec{g}_X \cdot \vec{t} z$ where g_X is the vector associated to X in the Feynman fan of a base triangulation.

[Arkani-Hamed, Frost, Salvatori, Plamondon, Thomas]

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Projective invariance of the Tr ϕ^3 scattering form means that

$$\mathcal{A}_n(z) \xrightarrow{g\text{-vector shift}} \mathcal{O}\left(\frac{1}{z^2}\right)$$

[Arkani-Hamed, Bai, He, Yan]

4-point Example

- Choose a triangulation of a square: X_{13}.
- Fill the rest of the basis up with c's: c_{13}.
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The other variables in this basis are

$$X_{24} = c_{13} - X_{13} \qquad\qquad c_{24} = c_{13}$$

Thus under the shift $X_{24} = X_{24} - z$ and the amplitude becomes

$$\hat{\mathcal{A}}_4 = \frac{1}{\hat{X}_{13}} + \frac{1}{\hat{X}_{24}} = \frac{X_{13} + X_{24}}{\hat{X}_{13}\hat{X}_{24}} \xrightarrow{z\to\infty} \frac{1}{z^2} c_{13}$$

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[Arkani-Hamed, Cao, Dong, Figueiredo, He]

Note that this is a 4-pion NLSM amplitude. At every n-point, there are special g-vector shifts that give rise to pions.

5-point Example

- Choose a triangulation of a pentagon: X_{13}, X_{14}.
- Fill the rest of the basis up with c's: c_{13}, c_{14}, c_{24}.
- Look at a specific shift $X_{13} \to X_{13} + z$, $X_{14} \to X_{14} - z$.

5-point Example

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- Fill the rest of the basis up with c's: c_{13}, c_{14}, c_{24}.
- Look at a specific shift $X_{13} \to X_{13} + z$, $X_{14} \to X_{14} - z$.

The other X's can be expressed in this basis:

$$X_{24} = c_{13} - X_{13} + X_{14}\,, \quad X_{25} = c_{13} + c_{14} - X_{13}\,, \quad X_{35} = c_{14} + c_{24} - X_{14}\,,$$
$$c_{25} = X_{25} + X_{13} - X_{35}\,, \quad c_{35} = X_{35} + X_{14} - X_{13}\,.$$

Thus under the shift:

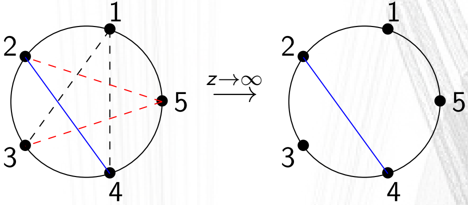
$$X_{24} \to X_{24}\,, X_{25} \to X_{25} - z\,, X_{35} \to X_{35} + z\,, c_{25} \to c_{25} - z\,, c_{35} \to c_{35} - z\,.$$

Taking $z \to \infty$, the amplitude is

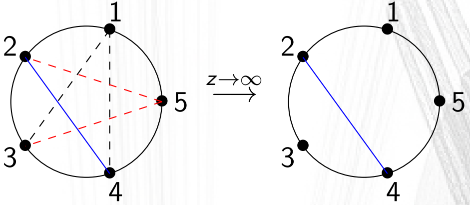
$$\mathcal{A}_5 \overset{z\to\infty}{\longrightarrow} \frac{1}{z^2}\frac{c_{14}}{X_{24}}$$

Can we predict this function?

Shifting the Surface



Shifting the Surface



Thus the function at infinity must look like:

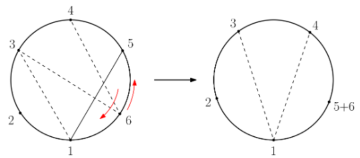
$$\frac{1}{z^2}\frac{\text{numerator of }(234)\times\text{numerator of }(1245)}{X_{24}}$$

- Dimensional analysis tells us that the numerator has dimension 1
- Kinematic surface tells us that the numerator factorizes
- Projective invariance of the scattering form on subsurfaces tells us the overall z scaling

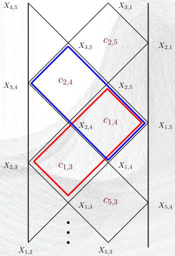
Assigning Numerators

Numerators can be assigned to subsurfaces via the following procedure:

- Base triangulation of subsurface is inherited from the full surface:



- Actual numerator can be read off from the kinematic mesh by drawing causal diamonds:



Final Formula

$$\boxed{\mathcal{A}_n^{\infty} = \prod_{\substack{S_i \subset S \\ |S_i| \geq 4}} \frac{1}{z^{|S_i|-2}} \prod_{S_i \subset S} \frac{1}{\mathrm{PT}_{S_i}} \sum_{\mathcal{T}_{\text{partial}}} \left(\prod_{i,j \in \mathcal{T}_{\text{partial}}} \frac{1}{X_{i,j}} \right) \prod_{\substack{S_i \subset S \\ n_i \geq 4}} c_{S_i}}$$

[SP, Skowronek, Spradlin, Volovich]

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Product of factors of $1/z$ for every subsurface larger than a triangle. Such surfaces involve a propagator that has been sent to infinity.

[SP, Skowronek, Spradlin, Volovich]

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Product of Parke-Taylor factors made out of components of the vector $\vec{t}$ that dictates the shift $X_{ij} \to X_{ij} + \vec{t}_{ij} z$.

$$\frac{1}{t_1(t_1 - t_2)(t_2 - t_3)\cdots(t_{n-4} - t_{n-3})t_{n-3}}$$

[SP, Skowronek, Spradlin, Volovich]

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Sum over all partial triangulations that contain unshifted chords.
This is the sum over all graphs that can be built from the propagators still available at infinity.

[SP, Skowronek, Spradlin, Volovich]

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Product of numerators associated to the various subsurfaces.

[SP, Skowronek, Spradlin, Volovich]

Properties

The result manifests the following properties

- Factorization: Functions at infinity $\mathcal{A}_n^\infty$ factorize into lower-point functions at infinity.
- Commutativity: The order of taking two subsequent g-vector shifts is immaterial.

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- Factorization: Functions at infinity $\mathcal{A}_n^\infty$ factorize into lower-point functions at infinity.
- Commutativity: The order of taking two subsequent g-vector shifts is immaterial.
- Pion amplitudes: For special g-vector shifts we get pion-scalar amplitudes. For example, consider a base triangulation (X_{13}, X_{14}, X_{15}) and a shift vector $\vec{t} = (1, 0, 0)$. This gives

$$\mathcal{A}_6 \overset{z\to\infty}{\longrightarrow} \frac{1}{z^2}\mathcal{A}_6[\pi_1\pi_2\phi_3\phi_4\phi_5\phi_6]$$

  For another example, consider a shift vector $\vec{t} = (1, 0, 1)$. This gives

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[Arkani-Hamed, Cao, Dong, Figueiredo, He]

Applications

1. A new positive geometry: A class of pion-scalar amplitudes are described by products of associahedra
2. Recursion relations: These manifest the zero and near-zero splitting properties of Tr ϕ^3, NLSM, YMS and special Galileon.
3. Cosmological wavefunction: The combinatorics of some graphs that contribute to flat-space coefficients are described by associahedra.

Applications

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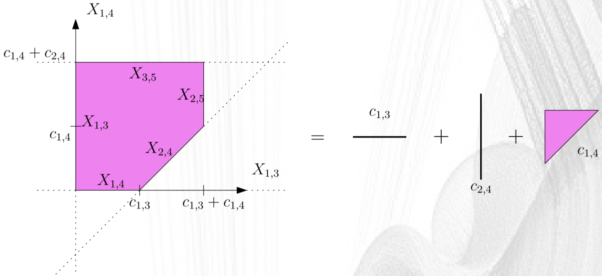
3. Cosmological wavefunction

Associahedron

Tr ϕ^3 amplitudes are given by the canonical function of the ABHY associahedron. At 5-point:



[Arkani-Hamed, Bai, He, Yan]

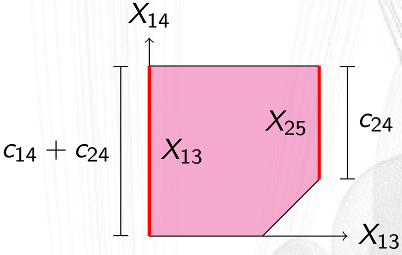
The canonical function is then

$$\frac{1}{X_{14}X_{24}}+\frac{1}{X_{24}X_{25}}+\frac{1}{X_{25}X_{35}}+\frac{1}{X_{13}X_{35}}+\frac{1}{X_{13}X_{14}}=\mathcal{A}_5$$

What happens when we take $X_{14}+z$, $X_{24}+z$, $X_{35}-z$ and $z\to\infty$?

Projecting the Associahedron

We have no access to the boundaries that have been shifted to infinity. All we have is



Since the lowest dimensional boundaries are no longer vertices, we have to include the lengths of the boundary components in the associated canonical function:

$$\frac{1}{z^2}\left(\frac{c_{14}+c_{24}}{X_{13}}+\frac{c_{24}}{X_{25}}\right)$$

A Special Case

In most cases, like the previous one, we end up with disjoint geometries i.e. a union of simplices.

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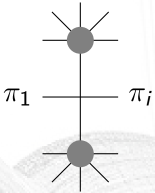
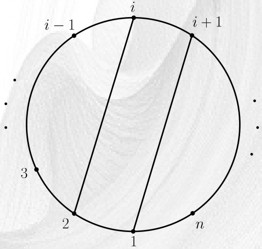
Yes! $\mathcal{A}_n(\pi\phi\cdots\phi\pi\phi\cdots\phi)$

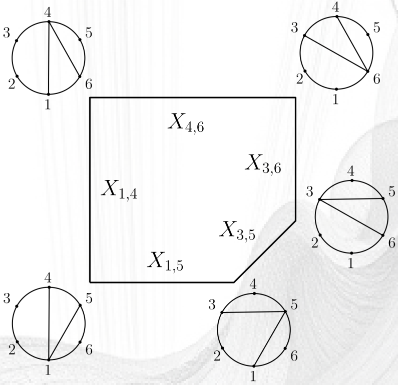
Feynman diagrams give us a good idea of why this is the case:





A New Positive Geometry

Thus a well-defined form on the $(n-1)$-point ABHY associahedron space gives the n-point $\mathcal{A}_n(\pi\pi\phi\cdots\phi)$ amplitudes.



Meanwhile for the non-adjacent case, we get a product of associahedra.

Applications

1. A new positive geometry
2. Recursion relations
3. Cosmological wavefunction

Recursion Relations

For any g-vector shift, the amplitudes vanish at infinity $\Rightarrow$ one can construct a recursion relation, similar to BCFW:

$$\mathcal{A}_n(z=0) = \sum_{ij} \operatorname*{Res}_{\hat{X}_{ij}=0} \frac{1}{z} \hat{\mathcal{A}}_n(z)$$

For a specific shift, we get a particularly nice set of recursion relations that manifests hidden zeros and near-zero splitting.

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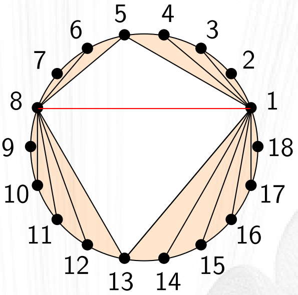
For a specific shift, we get a particularly nice set of recursion relations that manifests hidden zeros and near-zero splitting.
These amplitudes were shown to vanish on the support of

$$\begin{cases} s_{1m+1} = s_{1m+2} = \cdots = s_{1n-1} = 0 \\ s_{2m+1} = s_{2m+2} = \cdots = s_{2n-1} = 0 \\ \vdots \\ s_{m-1m+1} = \cdots = s_{m-1n-1} = 0 \end{cases}$$

and split smoothly when one of these conditions is relaxed.

Shift that Exposes the Zero

We use the shift that corresponds to



It turns out all but two of the residues vanish on the support of near-zero kinematics. This leaves us with the following recursion relation:

$$A_n[1,\ldots,n] = -\frac{1}{X_{1m+1}} \operatorname*{Res}_{z=X_B} \hat{A}_n(z) + \frac{1}{X_{mn}} \operatorname*{Res}_{z=X_T} \hat{A}_n(z) = \left(\frac{1}{X_{1m+1}} + \frac{1}{X_{mn}}\right) \operatorname*{Res}_{z=X_B} \hat{A}_n(z)$$

which is precisely the near-zero splitting theorem.

Multiple Splitting Formulae

On the support of near-zero kinematics:

Theory	Generic	Near-Zero	Explanation?
Tr ϕ^3	z^{-2}	z^{-2}	Projective Invariance
NLSM	z^0	z^{-2}	Commutativity of g-vector shifts
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Additionally, one can also get higher-order splitting formulae

$$A_n^{\phi^3}[1,\ldots,n] \xrightarrow[\{c_{1,j},c_{1,j+1},\ldots,c_{1,n-1}\}\neq 0]{\mathcal{Z}(X_{1,j})} A^{\phi^3}[1,\ldots,j]\Bigg\{\sum_{k=j}^{n-1}\frac{c_{1,k}}{X_{1,j}X_{j-1,n}}A^{\phi^3}[j-1,\ldots,n]$$

$$+\sum_{k=j}^{n-1}\sum_{l=j}^{k}\frac{c_{1,l}}{X_{1,j}X_{1,k+1}X_{j-1,k+1}}A^{\phi^3}[j-1,\ldots,k+1]\,A^{\phi^3}[k+1,\ldots,n,1]\Bigg\}.$$

[Jones, SP]

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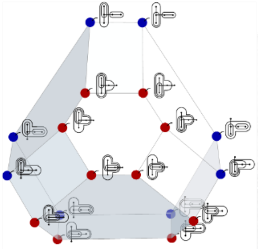
Tubings and Cosmological Wavefunction Coefficients

Observable: cosmological wavefunction graphs (or their integrands) of a conformally coupled scalar in flat space (or de Sitter).

Combinatorics: Given by graph tubings

Relation to Tr ϕ^3:

- The wfcs have hidden zeros, near-zero splitting and geometric descriptions.



[De, SP, Pokraka, Spradlin, Volovich]

- Are there corresponding g-vector shifts? Can we get theories with derivatives like NLSM?

Next Questions

- The function at infinity is fully characterized by considering the surface description of Tr ϕ^3.
- It always factorizes and is sometimes an amplitude.
- Projecting the associahedron gives a positive geometry for $\mathcal{A}_n[\pi\pi\phi\cdots\phi]$.
- Theories have zeros and split near them if and only if their residue at infinity vanishes.
- Many of these properties extend to cosmological wavefunction graphs.

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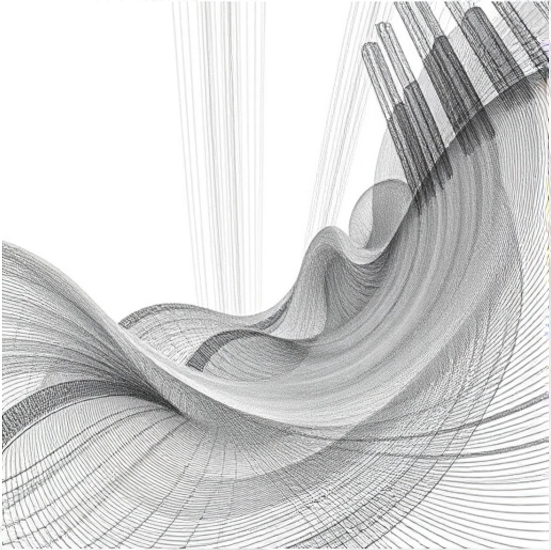
- Theories have zeros and split near them if and only if their residue at infinity vanishes.
- Many of these properties extend to cosmological wavefunction graphs.
- Is there a polytopal description of NLSM?
- Which other theories have vanishing residues at infinity?
- To what extent do cosmological observables exhibit surface descriptions?
- Can we use such properties to fix integrands at loop-level?



Thank You