

Loops in a Loop Expansion



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Based on [arXiv:2312.17736](https://arxiv.org/abs/2312.17736) with T Brown, U Oktem, J Trnka

WANTED

all-loop 4-point partial
amplitude in planar
 $\mathcal{N} = 4$ SYM

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Loops in a Loop Expansion

1. Why are we expanding?

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2. What are we expanding?

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Why are we expanding?

Can we resum perturbative results to calculate quantities at strong coupling?

$$\mathcal{A}_L = \int \mathcal{I}_L$$

e.g. $\mathcal{I}_3 = \left\{ \begin{array}{l} -\frac{s^3 t}{8} \text{ (diagram 1)} - \frac{s^2 t(l_1 + l_2)^2}{8} \text{ (diagram 2)} - \frac{s^2 t(l_1 + l_2)^2}{8} \text{ (diagram 3)} \\ -\frac{s t^3}{8} \text{ (diagram 4)} - \frac{s t^2(l_1 + l_2)^2}{8} \text{ (diagram 5)} - \frac{s t^2(l_1 + l_2)^2}{8} \text{ (diagram 6)} \end{array} \right\}$

[Bern, Dixon, Smirnov]

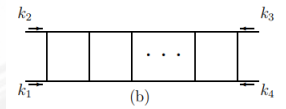
$\mathcal{I}_4 = \left\{ \begin{array}{l} \text{(b}_1\text{)} \quad \text{(b}_2\text{)} \quad \text{(b}_3\text{)} \quad \text{(b}_4\text{)} \quad \text{(c}_1\text{)} \quad \text{(d}_1\text{)} \\ \text{(d}_3\text{)} \quad \text{(d}_4\text{)} \quad \text{(d}_5\text{)} \quad \text{(e}_1\text{)} \quad \text{(e}_2\text{)} \quad \text{(e}_3\text{)} \\ \text{(e}_4\text{)} \quad \text{(e}_5\text{)} \quad \text{(e}_6\text{)} \quad \text{(g}_1\text{)} \end{array} \right\}$

[Bern, Czakon, Dixon, Kosower, Smirnov]

Resummation

Two examples of subsets of diagrams being resummed:

1. All ladders in ϕ^3



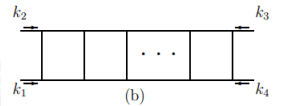
$$\xrightarrow{g \gg 1} \mathcal{O}(e^{-g\phi/(2\pi\sqrt{s})})$$

[Broadhurst, Davydychev]

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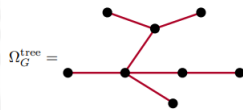


The diagram shows a ladder structure with four external legs labeled k_1, k_2, k_3, k_4 . The top and bottom horizontal lines are connected by three vertical rungs, with an ellipsis indicating more rungs. The diagram is labeled (b) below it.

$$\xrightarrow{g \gg 1} \mathcal{O}(e^{-g\phi/(2\pi\sqrt{s})})$$

[Broadhurst, Davydychev]

2. All “trees in loop space” in $\mathcal{N} = 4$ SYM



The diagram shows a tree structure with a central vertex connected to four other vertices. The vertices are represented by black dots, and the edges are red lines. The diagram is labeled $\Omega_G^{\text{tree}} =$ to its left.

[Arkani-Hamed, Henn, Trnka]

Both of these are **ad-hoc** selection of graphs to resum with different results. Does the resummation capture the strong coupling behaviour of the quantity of interest?

This Talk

Yes! ... in some examples.

Why did this subset capture strong coupling behaviour? Is this a good approximation? Can we go to the next order?

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What are we expanding?

At L -loop, the amplitude has divergences, $\mathcal{M}_L \sim \frac{1}{\epsilon^{2L}}$.

Divergences were shown to have special structure $\Rightarrow$ BDS Ansatz:

non-iterative terms

$$E_n^{(l)}(0) = 0$$



$$\mathcal{M}_n = \exp \left[\sum_{l=1}^{\infty} g^l \left(f^{(l)}(\epsilon) \mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right) \right]$$

for $n = 4, 5$

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$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$$

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Diagram illustrating the BDS Ansatz expansion:

- all- ϵ 1-loop amplitude** points to $\mathcal{M}_n^{(1)}(l\epsilon)$.
- non-iterative terms** points to $E_n^{(l)}(\epsilon)$, with the condition $E_n^{(l)}(0) = 0$.
- constant** points to $C^{(l)}$.
- cusplike anomalous dimension** points to $f_0^{(l)}$ in the expansion $f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$.

for $n = 4, 5$

IR Finite Quantities

BDS ansatz suggests we should look at

$$\log \mathcal{M}_4 = \frac{\Gamma_{\text{cusp}}}{\epsilon^2} + \mathcal{O}(\epsilon)$$

The divergence is milder. But is there an IR **finite** quantity we can compute instead?

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Consider the integrand of $\log \mathcal{M}_L = \int \tilde{\Omega}_L$.

Since each integration gives $\frac{1}{\epsilon^2}$, doing all but one integral gives an IR finite quantity that can only depend on

$$z = \frac{\langle AB12 \rangle \langle AB34 \rangle}{\langle AB14 \rangle \langle AB23 \rangle}$$

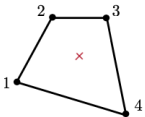
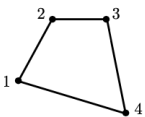
due to **dual conformal invariance** of the integrand.

A Third Loop

We denote:

$$\mathcal{F}_L(z) = \int_{\text{all but one}} \tilde{\Omega}_L \Rightarrow \mathcal{F}(g, z) = \sum_{L=1}^{\infty} \mathcal{F}_L(z) g^{2L}$$

Wilson loop $\leftrightarrow$ amplitude duality tells us that this quantity is


$$\mathcal{F}(g, z) = \frac{\text{Diagram with red 'x'}}{\text{Diagram without red 'x'}}$$


[Alday, Buchbinder, Tsyntlin, Engelund, Roiban, Henn, Sikorowski, Korchemsky, Mistlberger, Chicherin...]

This integrand $\mathcal{F}(g, z)$ can be used to obtain Γ_{cusp} .

This Talk

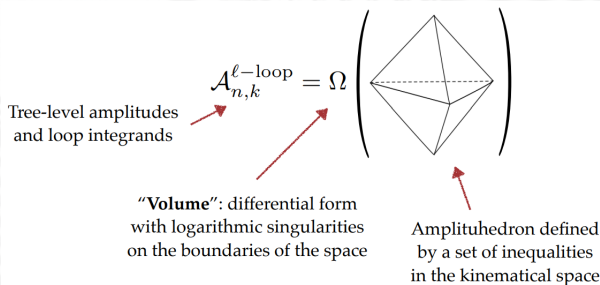
Can we get all-loop results beyond the “tree” approximation?

Can we capture $g \gg 1$ behaviour from the all-loop integrand given via the amplituhedron?

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How are we expanding?

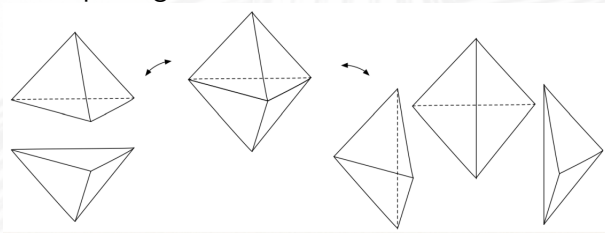
The amplituhedron is a polytope in twistor space whose volume gives tree amplitudes and loop integrands in $\mathcal{N} = 4$ SYM.



[Arkani-Hamed, Trnka]

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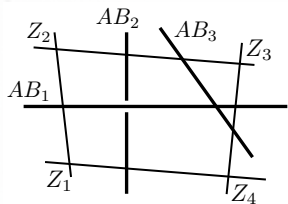


[Arkani-Hamed, Trnka]

In practice, calculating this volume form involves **triangulation** which is non-trivial. We take an algebraic approach.

Four-Point Amplituhedron

At L -loop, introduce L lines in momentum twistor space,



$Z_i \rightarrow$ External momentum twistors

$AB_i \rightarrow$ Loop lines

Z_i satisfy $\langle 1234 \rangle > 0$, while AB_i satisfy the following conditions:

$$\langle AB_i aa+1 \rangle > 0 \quad \langle AB_i 13 \rangle < 0 \quad \langle AB_i 24 \rangle < 0 \quad \langle AB_i AB_j \rangle > 0$$

We are looking for a canonical form on the space carved out by these inequalities that has **logarithmic singularities** on the boundaries.

Form of the Form

The first set of inequalities reduce to

$$x_i, y_i, w_i, z_i > 0$$

$$\text{where we use } Z_{A_i} = Z_1 + x_i Z_2 - y_i Z_4, \quad Z_{B_i} = Z_3 + w_i Z_2 + z_i Z_4.$$

This part of the form is easy,

$$\Omega_L \sim \prod_{i=1}^L \frac{dx_i}{x_i} \frac{dy_i}{y_i} \frac{dz_i}{z_i} \frac{dw_i}{w_i}$$

The mutual conditions $\langle AB_i AB_j \rangle > 0$ are quadratic and harder to see. We can work simplex by simplex, e.g.

$$-(x_2 - x_1)(w_2 - w_1) - (y_2 - y_1)(z_2 - z_1) > 0 \Rightarrow w_2 > w_1 + \frac{(y_1 - y_2)(z_1 - z_2)}{(x_1 - x_2)}$$

Form of the Form

This in a particular simplex

$$x_1 > x_2, y_1 > y_2, z_1 > z_2, w_2 > w_1$$

$$\Omega_2^{(1)} \supset \frac{dw_2}{w_2 - w_1 - \frac{(y_1 - y_2)(z_1 - z_2)}{(x_1 - x_2)}}.$$

Adding up the contributions,

$$\Omega_2 = \sum_{i=1}^8 \Omega_2^{(i)}$$

This increases quickly in complexity.

Instead we will try to fix the form using **cut conditions** directly in AB_i variables:

$$\Omega_L = \int \frac{\mathcal{N}_L}{\mathcal{D}_L} d\mu_L \quad d\mu_L = \prod_{i=1}^L \langle AB_i d^2 A_i \rangle \langle AB_i d^2 B_i \rangle$$

Numerators

1-Loop:

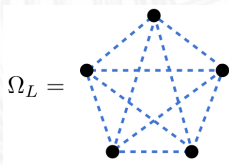
$$\frac{\langle 1234 \rangle^2 d\mu}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle} = \frac{dx_1}{x_1} \frac{dy_1}{y_1} \frac{dz_1}{z_1} \frac{dw_1}{w_1}$$

2-Loop: Now more complicated in x, y, z, w , involving mutual conditions.

$$\frac{\mathcal{N}_2 \langle 1234 \rangle^2 d\mu}{\langle AB_1 AB_2 \rangle \prod_{i=1}^2 \langle AB_i 12 \rangle \langle AB_i 23 \rangle \langle AB_i 34 \rangle \langle AB_i 14 \rangle} \neq \prod_i \frac{dx_i}{x_i} \frac{dy_i}{y_i} \frac{dz_i}{z_i} \frac{dw_i}{w_i}$$

3-Loop: Even more complicated...

We can represent these conditions via a graph:



Log Form

We would like $\log \mathcal{M}_L = \int \tilde{\Omega}_L$ where

$$\tilde{\Omega}_1 = \Omega_1(AB)$$

$$\tilde{\Omega}_2 = \Omega_2(AB, CD) - \Omega_1(AB)\Omega_1(CD)$$

$$\begin{aligned}\tilde{\Omega}_3 = & \Omega_3(AB, CD, EF) - \Omega_2(AB, CD)\Omega_1(EF) - \text{cyc} \\ & + 2\Omega_1(AB)\Omega_1(CD)\Omega_1(EF)\end{aligned}$$

$$\vdots$$

Are these canonical forms on a different space?

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Are these canonical forms on a different space? **Yes!**

[Arkani-Hamed, Henn, Trnka]

Negative Geometries

Use the relation:

$$\begin{array}{c} \bullet \text{---} \bullet \\ AB_1 \quad AB_2 \\ \langle AB_1 AB_2 \rangle > 0 \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ AB_1 \quad AB_2 \\ \langle AB_1 AB_2 \rangle < 0 \end{array} = \begin{array}{c} \bullet \quad \bullet \\ AB_1 \quad AB_2 \\ \text{no sign restriction} \end{array}$$

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To express the amplituhedron in loop space in terms of these negative links:

$$\Omega_L = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \sum_{\text{all } G} (-1)^{E(G)} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

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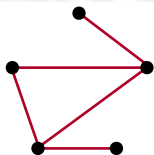
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A trick familiar to us from Feynman graphs gives

$$\sum_{\text{all } G} (-1)^{E(G)} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \exp \left(\sum_G (-1)^{E(G)} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right)$$

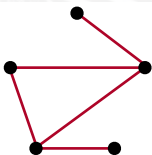
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So we get the expansion:

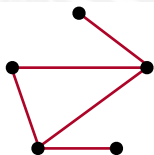
$$\mathcal{F}(g, z) = \bigotimes - (g^2) \bigotimes \text{---} \bullet$$

$$+ (g^2)^2 \left\{ \bigotimes \text{---} \bullet \text{---} \bullet + \frac{1}{2!} \bigotimes \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} - \frac{1}{2!} \bigotimes \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \right\}$$

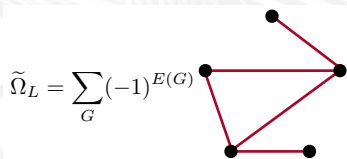
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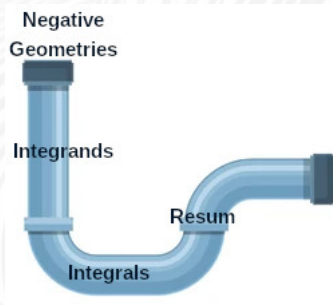
$$\tilde{\Omega}_L = \sum_G (-1)^{E(G)}$$



Negative Geometries



Recall our initial motivation was to get strong coupling results.



$$\mathcal{F}(g, z)$$
$$\Gamma_{\text{cusp}}(g)$$

Ladder and Tree Resummation

Results at strong coupling:

$$\begin{aligned}\mathcal{F}^{\text{ladder}}(g, z) &\xrightarrow{g \gg 1} 0, & \Gamma^{\text{ladder}}(g) &\xrightarrow{g \gg 1} 4\sqrt{2}g \\ \mathcal{F}^{\text{tree}}(g, z) &\xrightarrow{g \gg 1} \frac{-z}{(1+z)^2}, & \Gamma^{\text{tree}}(g) &\xrightarrow{g \gg 1} \frac{8}{\pi}g \\ \mathcal{F}^{\text{true}}(g, z) &\xrightarrow{g \gg 1} g \frac{z(2(1-z) + (z+1)\log z)}{(z-1)^3}, & \Gamma^{\text{true}}(g) &\xrightarrow{g \gg 1} 2g\end{aligned}$$

[Benna, Benvenuti, Klebanov, Scardicchio] [Basso, Korchemsky, Kotanski]

[Alday, Buchbinder, Tsytlín][Beisert, Eden, Staudacher][Arkani-Hamed, Henn, Trnka]

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At weak coupling, we compare this to known results from integrability:

$r_{\text{ladder/cusp}} =$	g^2	g^4	g^6	g^8	g^{10}	g^{12}	g^{14}	$\dots$
	1	1	0.73	0.44	0.25	0.14	0.07	$\dots$
$r_{\text{cusp/tree}} =$	g^2	g^4	g^6	g^8	g^{10}	g^{12}	g^{14}	$\dots$
	1	1	0.92	0.83	0.74	0.63	0.53	$\dots$

Does 1-loop in loop space bring us closer? **Our aim!**

Fixing Integrands

Denominator: Fixed by pole structure

$$\frac{1}{\prod_{i=1}^L \prod_{a=1}^4 \langle AB_i aa + 1 \rangle} \times \frac{1}{\prod_{ij} \langle AB_i AB_j \rangle}$$

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Numerator: Must vanish on unphysical cuts:

1. Cuts that violate the **positivity or negativity** of the links or vertices:
Involve one link i.e. two vertices at a time.
2. Cuts that expose a **double pole**: Involve all vertices in a cycle.

Second cut is irrelevant in the tree case and as a result, the tree numerator can be built link by link.

Examples of Cuts

Positive cuts: A cut that violates $\langle ABCD \rangle > 0$, such as

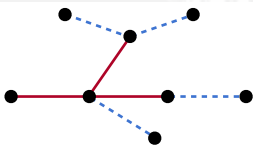
$$\begin{aligned}\langle AB12 \rangle &= \langle AB34 \rangle = \langle CD23 \rangle = \langle CD14 \rangle = 0 \\ \Rightarrow x_2 &= y_1 = z_1 = w_2 = 0 \\ \Rightarrow \langle ABCD \rangle &= -x_1 w_1 - y_2 z_2 < 0\end{aligned}$$

Negative cuts: A cut that violates $\langle ABCD \rangle < 0$, such as

$$\begin{aligned}\langle AB12 \rangle &= \langle AB14 \rangle = \langle CD23 \rangle = \langle CD34 \rangle = 0 \\ \Rightarrow x_1 &= y_1 = z_2 = w_2 = 0 \\ \Rightarrow \langle ABCD \rangle &= x_1 w_2 + y_1 z_2 > 0\end{aligned}$$

Depending on the type of link, we will require that the numerator vanishes on either type of cut.

Tree Result



$$\Omega_G^{\text{tree}} = \frac{d\mu \mathcal{N}_G}{\prod_i D_i \cdot \prod_{\text{links}} \langle AB_i AB_j \rangle}$$

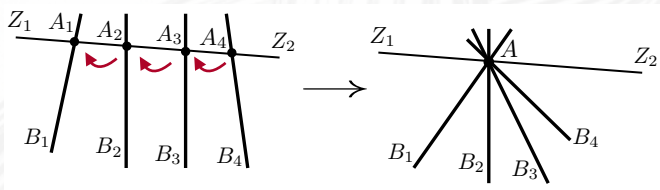
$$\mathcal{N}_G = \langle 1234 \rangle^{L+1} \times \prod_{\text{pos links}} N_{ij}^{(+)} \times \prod_{\text{neg links}} N_{ij}^{(-)}$$

$$N_{ij}^{(+)} \equiv \langle AB_i 12 \rangle \langle AB_j 34 \rangle + \text{cyc}$$

$$N_{ij}^{(-)} \equiv \langle AB_i 13 \rangle \langle AB_j 24 \rangle + \langle AB_i 24 \rangle \langle AB_j 13 \rangle$$

Example of Cuts

Double pole cuts: Once there is a cycle, $\langle AB_i 12 \rangle = 0$ and $\langle AB_i AB_{i+1} \rangle = 0$ forces $\langle AB_1 AB_L \rangle = 0$. Pictorially,



Crucially, $\mathcal{N}_{\text{tree}}$ vanishes on “positive” and “negative” cuts respectively, but not on double pole cuts.

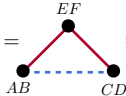
So we need to introduce a remainder that **cancels** out the contribution of $\mathcal{N}_{\text{tree}}$ on double pole cuts.

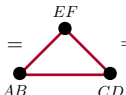
Remainders at 1-Loop

We decompose the 1-loop numerators as

$$\prod_{\text{pos links}} N_{ij}^{(+)} \times \prod_{\text{neg links}} N_{ij}^{(-)} + R_L$$

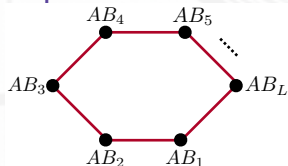
This same remainder function (up to sign) appears in different loops:

$$\Omega_3^{(+-)} = \frac{N_{12}^{(+)} N_{23}^{(-)} N_{13}^{(-)} + R_{123}}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle \langle ABCD \rangle \langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD14 \rangle \langle CDEF \rangle \langle EF12 \rangle \langle EF23 \rangle \langle EF34 \rangle \langle EF14 \rangle \langle ABEF \rangle}$$


$$\Omega_3^{(---)} = \frac{N_{12}^{(-)} N_{23}^{(-)} N_{13}^{(-)} - R_{123}}{\langle AB12 \rangle \langle AB23 \rangle \langle AB34 \rangle \langle AB14 \rangle \langle ABCD \rangle \langle CD12 \rangle \langle CD23 \rangle \langle CD34 \rangle \langle CD14 \rangle \langle CDEF \rangle \langle EF12 \rangle \langle EF23 \rangle \langle EF34 \rangle \langle EF14 \rangle \langle ABEF \rangle}$$


Thus R must vanish on **both** “positive” and “negative” cuts $\Rightarrow$ it must be built out of a certain building blocks.

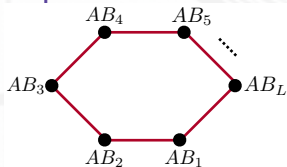
1-Loop Result



$$\mathcal{N}_G = \prod_{i=1}^L N_{i,i+1}^{(-)} + (-1)^L R_L^{1-\text{loop}}$$

$$R_L^{1-\text{loop}} = \left\{ (2^L - 4) N_1^{(a)} N_2^{(a)} \dots N_L^{(a)} \right. \\ \left. - \sum_{p \in \text{even}} \sum_i \{ N_{i_1 i_2 \dots i_p}^{(a)} N_{i_1 i_2 \dots i_p}^{(c)} N_{i_p+1}^{(a)} \dots N_{i_L}^{(a)} \} \right\} + (a \rightarrow b)$$

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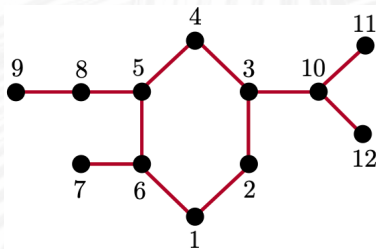
$$N_{12 \dots p}^{(a)} = \prod_{j \in \text{odd}} \langle AB_j 12 \rangle \prod_{k \in \text{even}} \langle AB_k 34 \rangle + \prod_{j \in \text{odd}} \langle AB_j 34 \rangle \prod_{k \in \text{even}} \langle AB_k 12 \rangle$$

$$N_{12 \dots p}^{(b)} = \prod_{j \in \text{odd}} \langle AB_j 23 \rangle \prod_{k \in \text{even}} \langle AB_k 14 \rangle + \prod_{j \in \text{odd}} \langle AB_j 14 \rangle \prod_{k \in \text{even}} \langle AB_k 23 \rangle$$

$$N_{12 \dots p}^{(c)} = \prod_{j \in \text{odd}} \langle AB_j 13 \rangle \prod_{k \in \text{even}} \langle AB_k 24 \rangle + \prod_{j \in \text{odd}} \langle AB_j 24 \rangle \prod_{k \in \text{even}} \langle AB_k 13 \rangle$$

Adding Branches

Since branches do not add any new double pole conditions, it is simple to guess what the result is for a generic configuration.



$$\mathcal{N}_G = \prod_{\text{links}} N_{ij}^{(-)} + (-1)^{L'} \prod_{\text{branches}} N_{ij}^{(-)} \times R_{L'}^{1-\text{loop}}$$

Loop Decomposition

The 1-loop answer split into two parts:

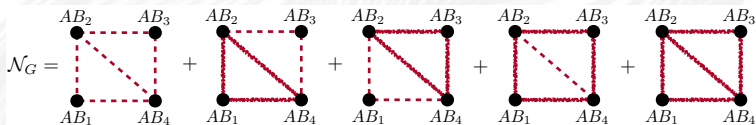
1. Part that vanishes on all “tree” cuts
1. Part that is required for “1-loop” cut

Loop Decomposition

The 1-loop answer split into two parts:

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This inspires the following general decomposition:

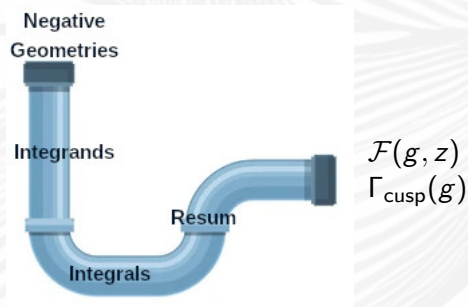


There is only one unknown piece which is

$$R_{1234,13}^{2-\text{loop}} = 0 \quad \text{on any cut when } \langle AB_i AB_j \rangle \text{ has a definite sign}$$

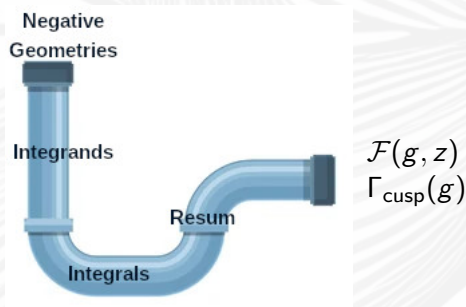
Moving Forward

Ideally, we have the pipeline:



Moving Forward

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But for generic cycles, the integrated result is **not** known.

The 3-loop cycle (i.e. the triangle) is known. We can use differential equations to find triangle + branches.

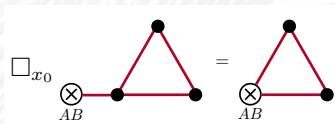
Even in this case, resummation is unclear...

Using Differential Equations

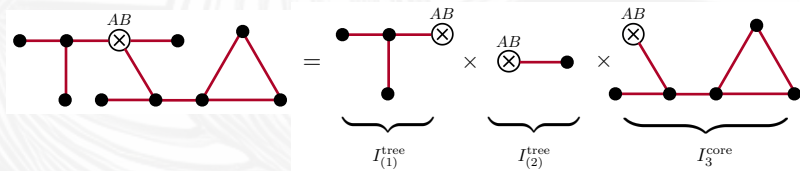
1. Action of Laplace operator

$$\square_{x_0} \frac{1}{(x - x_0)^2} = -4\pi^2 i \delta^4(x - x_0)$$

On the integrand (rewritten in some appropriate way), this collapses a propagator:



2. Factorization occurs around the unintegrated loop:



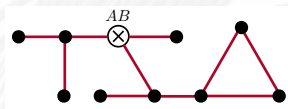
Triangles and Friends

We can invert this Laplacian equation by assuming:

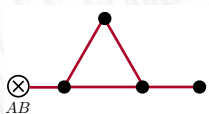
1. Result can be written in terms of HPLs
2. $\mathcal{F}(z \rightarrow -1) = 0$
3. $\mathcal{F}(\frac{1}{z}) = \mathcal{F}(z)$

The last two are consequences of the structure of the integrand.

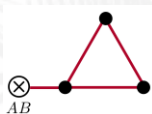
This gives us all integrals of the type:



But not those of the type:



Integrated Results: An Example



$$\begin{aligned} I_{3,1}(z) = & 8H_{0,0,0,0,0,0}(z) + 8H_{0,0,-1,0,0,0}(z) - 16H_{0,0,-1,-1,0,0}(z) \\ & + 8H_{0,0,-2,0,0}(z) - 8\zeta_3(2H_{0,0,-1}(z) - H_{0,0,0}(z)) \\ & + 4\pi^2(H_{0,0,-1,0}(z) - H_{0,0,-1,-1}(z) + H_{0,0,-2}(z)) + \frac{13\pi^4}{90} \log^2(z) \\ & + C_{3,1} \log(z) + D_{3,1}, \end{aligned}$$

To fix the constants that, we impose the boundary conditions

$$C_{3,1} = -16\zeta_{3,2} + 12\pi^2\zeta_3 - 80\zeta_5$$

$$D_{3,1} = 16\zeta_{3,3} - 16\zeta_3^2 + \frac{289\pi^6}{1890}.$$

Extracting Γ_{cusp}

We have another integral to perform,

$$g \frac{\partial}{\partial g} \Gamma_{\text{cusp}} = \oint F(z) dz$$

Integrating gives,

	g^2	g^4	g^6	g^8	g^{10}	g^{12}	g^{14}
$\Gamma_{\text{tree}}/\Gamma_{\text{true}}$	1	1	1.091	1.198	1.353	1.579	1.893
$\Gamma_{\Delta,n}/\Gamma_{\text{true}}$	0	0	-0.091	-0.053	-0.059	-0.074	-0.097
$\Gamma_{\text{tree},\Delta,n}/\Gamma_{\text{true}}$	1	1	1	1.145	1.294	1.505	1.796

Exact Results from Differential Equations

The ladder and tree results did not come from explicit resummation, but rather by solving differential equations:

$$\begin{aligned}\frac{1}{2}(z\partial_z)^2 \mathcal{F}^{\text{ladder}} + g^2 \mathcal{F}^{\text{ladder}} &= 0 \\ \frac{1}{2}(z\partial_z)^2 \log \mathcal{F}^{\text{tree}} + g^2 \mathcal{F}^{\text{tree}} &= 0\end{aligned}$$

A similar equation can be set up for the triangle.

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$$g \frac{\partial}{\partial g} \Gamma_{\text{cusp}} = \oint F_{\triangle}(z) F_{\text{ladder}}(z) dz$$

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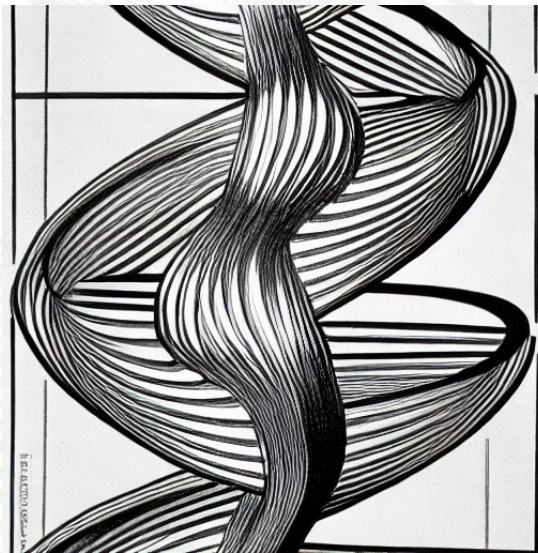
$$g \frac{\partial}{\partial g} \Gamma_{\text{cusp}} = \oint F_{\Delta}(z) F_{\text{ladder}}(z) dz$$



Future Directions

- ▶ Can these contributions be resummed?
- ▶ Can the other 1- or higher-cycle integrands be integrated?
- ▶ What other quantities can we use similar approximations for?
- ▶ Which contributions/how many cycles capture $\mathcal{F}(g, z)$ at strong coupling?
- ▶ Do these results have nice structure at the symbol level?

⋮



Thank You