Negative Geometries and $\mathcal{N} = 4$ SYM

Shruti Paranjape
Brown University

Indian Strings Meeting 2025

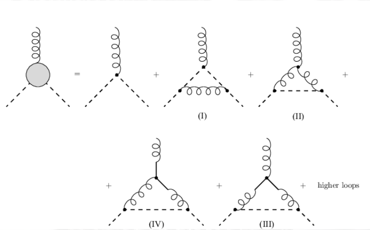
Based on arXiv:2312.17736 with T Brown, U Oktem, J Trnka
and arXiv:2512.xxxx with L Dixon, U Oktem, J Trnka, S-Q Zhang, Y Xu

Constructing Scattering Amplitudes

Usual method:

1. Write down Lagrangian $\mathcal{L}$ - easy
2. Extract Feynman rules - sort of easy
3. Draw all Feynman diagrams - sort of hard
4. Calculate all diagrams and sum - hard

The calculation is organized in a perturbative series:



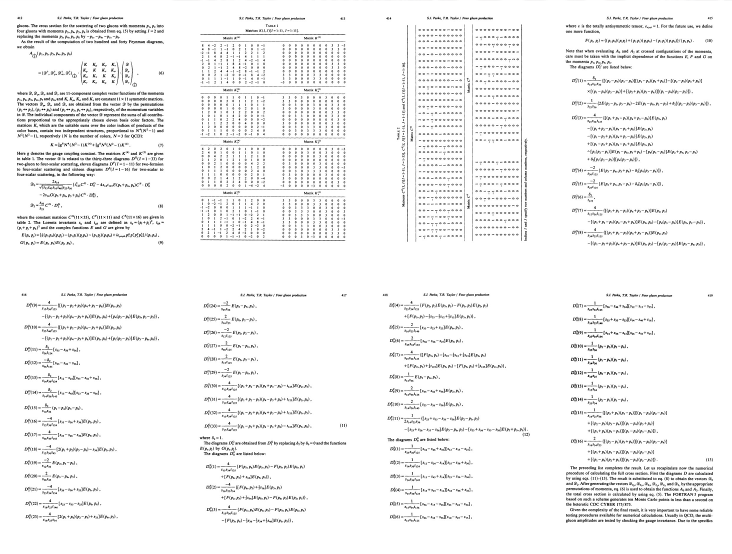
Example: Gluon Amplitudes

$$\mathcal{V}_{\mathrm{QCD}} \sim g\ f^{abc}\partial^{\mu}A^{a}_{\mu}A^{b}_{\nu}A^{c\nu} + g^2 f^{abc}f^{ade}A^{a}_{\mu}A^{b}_{\nu}A^{d\mu}A^{e\nu}$$

In QCD: $g(p_1)g(p_2) \to g(p_3)g(p_4)g(p_5)g(p_6)$

Approximately 220 Feynman diagrams:

Glimpse at the Calculations by Parke and Taylor (1985)



Feynman Doesn't Rule

Parke-Taylor formula (1986):

$$\mathcal{A}_n(1^-2^-3^+\cdots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,n\rangle\langle n1\rangle}$$

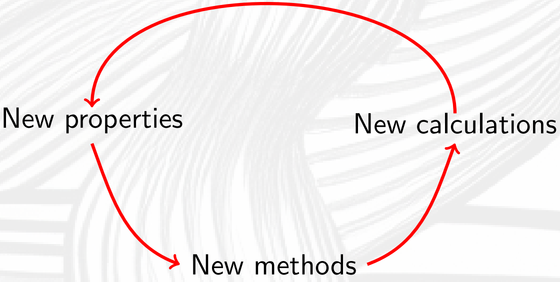
Why is this simple?

Feynman Doesn't Rule

Parke-Taylor formula (1986):

$$\mathcal{A}_n(1^-2^-3^+\cdots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,n\rangle\langle n1\rangle}$$

Why is this simple? Symmetry

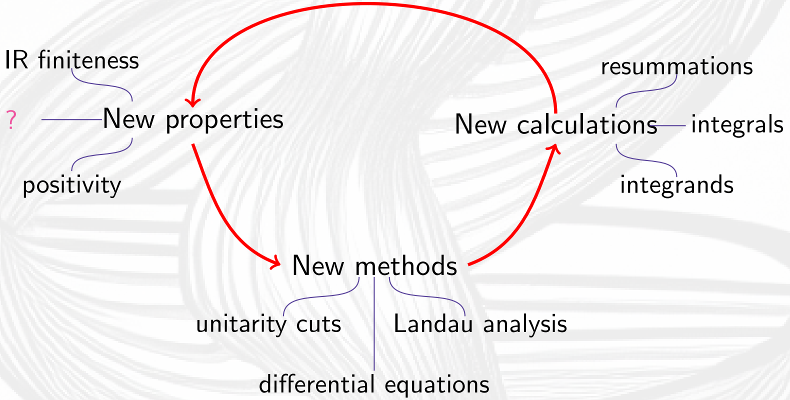


Feynman Doesn't Rule

Parke-Taylor formula (1986):

$$\mathcal{A}_n(1^-2^-3^+\cdots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n-1,n\rangle\langle n1\rangle}$$

Why is this simple? Symmetry



$\mathcal{N} = 4$ SYM

- Color-ordered amplitudes have much fewer singularities.
- Tree amplitudes are constructible recursively.

  [Britto, Cachazo, Feng, Witten]
- It has dual conformal symmetry.

  [Drummond, Korchemsky, Sokatchev]
- It has a holographic dual which one can check answers against or make predictions for.

  [Maldacena]
- There exists a polytope description of its Feynman integrands (amplituhedron).

  [Arkani-Hamed, Trnka]
- The integrals lead to a space of functions with a well-defined symbol.

  [Goncharov, Spradlin, Vergu, Volovich]
- Symbol bootstrap via properties such as cluster adjacency.

  [Drummond, Foster, Gürdoğan]

$\vdots$

Negative Geometries: Loops in a Loop Expansion

1. Why are we expanding?

Negative Geometries: Loops in a Loop Expansion

1. Why are we expanding?
2. What are we expanding?

Negative Geometries: Loops in a Loop Expansion

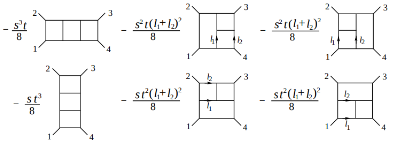
1. Why are we expanding?
2. What are we expanding?
3. How are we expanding?

Why are we expanding?

Lots of new high-loop results available:

$$\mathcal{A}_L = \int \mathcal{I}_L$$

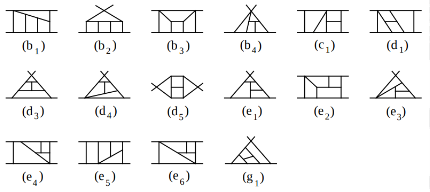
e.g. $\mathcal{I}_3 = \Bigg\{$



$\Bigg\}$

[Bern, Dixon, Smirnov]

$\mathcal{I}_4 = \Bigg\{$



$\Bigg\}$

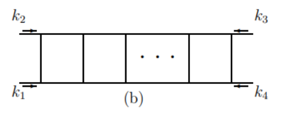
[Bern, Czakon, Dixon, Kosower, Smirnov]

Can we resum perturbative results to calculate quantities at strong coupling?

Resummation

Two examples of subsets of diagrams being resummed[1]:

1. All ladders in ϕ^3 (see Siddharth's talk!)



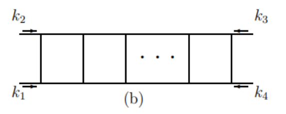
[Broadhurst, Davydychev]

[1] There are important examples of resummations in specific limits such as $\hbar \to 0$.

Resummation

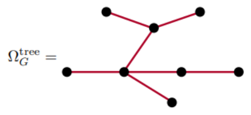
Two examples of subsets of diagrams being resummed[1]:

1. All ladders in ϕ^3 (see Siddharth's talk!)



[Broadhurst, Davydychev]

2. All "trees in loop space" in $\mathcal{N} = 4$ SYM



[Arkani-Hamed, Henn, Trnka]

Both of these are ad-hoc selection of graphs to resum with different results. Does the resummation capture the strong coupling behaviour of the quantity of interest?

[1] There are important examples of resummations in specific limits such as $\hbar \to 0$.

This Talk

Yes! ... in some examples.

Why did this subset capture strong coupling behaviour? Is this a good approximation? Can we go to the next order?

This talk: Loops in a loop expansion

This Talk

Yes! ... in some examples.

Why did this subset capture strong coupling behaviour? Is this a good approximation? Can we go to the next order?

This talk: Loops in a loop expansion

1. Why are we expanding? ✓
2. What are we expanding?
3. How are we expanding?

What are we expanding?

At L-loop, the amplitude has divergences, $\mathcal{M}^{(L)} \sim \frac{1}{\epsilon^{2L}}$.

Divergences were shown to have special structure $\Rightarrow$ BDS Ansatz:

non-iterative terms

$E_n^{(l)}(0) = 0$

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} g^l \left(f^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

for $n = 4, 5$

What are we expanding?

At L-loop, the amplitude has divergences, $\mathcal{M}^{(L)} \sim \frac{1}{\epsilon^{2L}}$.

Divergences were shown to have special structure $\Rightarrow$ BDS Ansatz:

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} g^l \left(f^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) + \underbrace{C^{(l)}}_{\text{constant}} + \overbrace{E_n^{(l)}(\epsilon)}^{\substack{\text{non-iterative terms} \\ E_n^{(l)}(0) = 0}} \right)\right]$$

for $n = 4, 5$

What are we expanding?

At L-loop, the amplitude has divergences, $\mathcal{M}^{(L)} \sim \frac{1}{\epsilon^{2L}}$.

Divergences were shown to have special structure $\Rightarrow$ BDS Ansatz:

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} g^l \left(f^{(l)}(\epsilon) \overbrace{\mathcal{M}_n^{(1)}(l\epsilon)}^{\text{all-}\epsilon\text{ 1-loop amplitude}} + \underbrace{C^{(l)}}_{\text{constant}} + \overbrace{E_n^{(l)}(\epsilon)}^{\substack{\text{non-iterative terms} \\ E_n^{(l)}(0)=0}} \right)\right]$$

for $n = 4, 5$

What are we expanding?

At L-loop, the amplitude has divergences, $\mathcal{M}^{(L)} \sim \frac{1}{\epsilon^{2L}}$.

Divergences were shown to have special structure $\Rightarrow$ BDS Ansatz:

non-iterative terms

$E_n^{(l)}(0) = 0$

all-ϵ 1-loop amplitude

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} g^l \left(f^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$

constant

for $n = 4, 5$

What are we expanding?

At L-loop, the amplitude has divergences, $\mathcal{M}^{(L)} \sim \frac{1}{\epsilon^{2L}}$.

Divergences were shown to have special structure $\Rightarrow$ BDS Ansatz:

non-iterative terms

$E_n^{(l)}(0) = 0$

all-ϵ 1-loop amplitude

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} g^l \left(f^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$$

constant

cusp anomalous dimension

for $n = 4, 5$

IR Finite Quantities

BDS ansatz suggests we should look at

$$\log \mathcal{M}_4 = \frac{\Gamma_{\text{cusp}} \mathcal{M}^{(1)}_{(-2)}}{\epsilon^2} + \mathcal{O}(\epsilon)$$

The divergence is milder. But is there an IR finite quantity we can compute instead?

IR Finite Quantities

BDS ansatz suggests we should look at

$$\log \mathcal{M}_4 = \frac{\Gamma_{\text{cusp}} \mathcal{M}^{(1)}_{(-2)}}{\epsilon^2} + \mathcal{O}(\epsilon)$$

The divergence is milder. But is there an IR finite quantity we can compute instead?

Consider the integrand of $\log \mathcal{M}_4^{(L)} = \int \tilde{\Omega}_L$.

Since each integration gives $\frac{1}{\epsilon^2}$, doing all but one integral gives an IR finite quantity that can only depend on

$$z = \frac{\langle AB12 \rangle \langle AB34 \rangle}{\langle AB14 \rangle \langle AB23 \rangle}$$

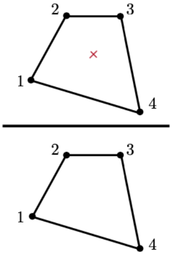
due to dual conformal invariance of the integrand.

A Third Loop

We denote:

$$\mathcal{F}_L(z) = \int_{\text{all but one}} \tilde{\Omega}_L \quad \Rightarrow \quad \mathcal{F}(g,z) = \sum_{L=1}^{\infty} \mathcal{F}_L(z) g^{2L}$$

Wilson loop $\leftrightarrow$ amplitude duality tells us that this quantity is

$\mathcal{F}(g,z) =$ 

[Alday, Buchbinder, Tsytelin, Engelund, Roiban, Henn, Sikorowski, Korchemsky, Mistlberger, Chicherin...]

This integrand $\mathcal{F}(g,z)$ can be used to obtain Γ_{cusp}.

Known Asymptotic Behaviour

Results from integrability, but no strong coupling results from amplitudes:

[Beisert, Eden, Staudacher]

$$\mathcal{F}(g,z) \to \begin{cases} -g^2 + g^4(\log^2 z + \pi^2) + \cdots & g \ll 1 \\ g\frac{z}{(z-1)^3}\left(2(1-z) + (z+1)\log z\right) + \cdots & g \gg 1 \end{cases}$$

As we saw this is related to a particular ratio of Wilson loops.

[Alday, Henn, Sirkowski] [Henn, Korchemsky, Mistlberger]

$$\Gamma(g) \to \begin{cases} 4g^2 - 8\zeta_2 g^4 + \cdots & g \ll 1 \\ 2g - \frac{3\log 2}{2\pi} + \cdots & g \gg 1 \end{cases}$$

Perturbative regime results can be reproduced from amplitudes.

[Bern, Czakon, Dixon, Kosower, Smirnov]

Strong coupling regime results can be reproduced from minimal surface anchored to the boundary at a null polygon.

[Benna, Benvenuti, Klebanov, Scardicchio] [Basso,Korchemsky, Kotanski]

[Alday, Buchbinder, Tsytelin]

This Talk

Can we get all-loop results for $\mathcal{F}(z)$ and Γ_{cusp}?

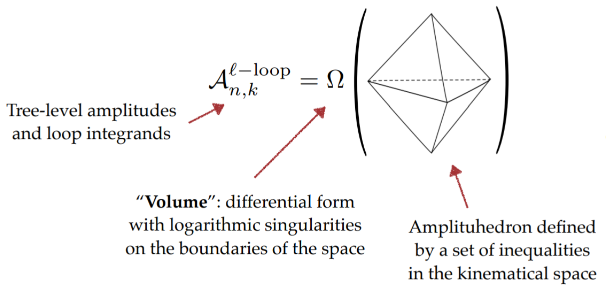
Can we capture $g \gg 1$ behaviour from the all-loop integrand given via the amplituhedron?

Can we better understand the function space of $\mathcal{F}(z)$ and the number space of Γ_{cusp}?

1. Why are we expanding? ✓
2. What are we expanding? ✓
3. How are we expanding?

How are we expanding?

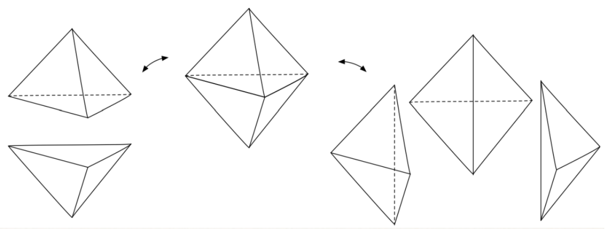
The amplituhedron is a polytope in twistor space whose volume gives tree amplitudes and loop integrands in $\mathcal{N} = 4$ SYM.



[Arkani-Hamed, Trnka]

How are we expanding?

The amplituhedron is a polytope in twistor space whose volume gives tree amplitudes and loop integrands in $\mathcal{N} = 4$ SYM.



[Arkani-Hamed, Trnka]

In practice, calculating this volume form involves triangulation which is non-trivial. We take an algebraic approach.

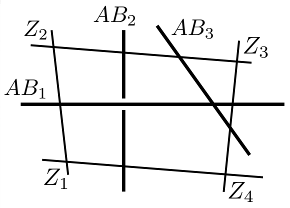
Quick Recap of Momentum Twistors

For dual conformal quantities, it is useful to use (super-)momentum twistors Z^I. Important to remember:

- $I = 1, \cdots, 4$, but Z^I is projective.
- Any choice of Z's will obey momentum conservation.
- One point in Z space corresponds to one external momentum.
- $SU(2,2)$ invariants are given by 4-brackets i.e. determinants $\langle 1234 \rangle = \epsilon^{IJKL} Z^1_I Z^2_J Z^3_K Z^4_L$.

Four-Point Amplituhedron*

At L-loop, introduce L lines in momentum twistor space,



$Z_i \to$ External momentum twistors
$AB_i \to$ Loop lines

Z_i satisfy $\langle 1234 \rangle > 0$, while AB_i satisfy the following conditions:

$$\langle AB_i a a{+}1 \rangle > 0 \quad \langle AB_i 13 \rangle < 0 \quad \langle AB_i 24 \rangle < 0 \quad \langle AB_i AB_j \rangle < 0$$

We are looking for a canonical form on the space carved out by these inequalities i.e. a form that has logarithmic singularities on all the boundaries and at no other location.

Numerators

1-Loop:

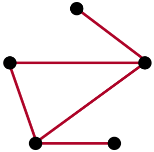
$$\frac{\langle 1234\rangle^2 d\mu}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle}$$

2-Loop: Now has a non-trivial numerator that ensures that unphysical cuts have vanishing residues..

$$\frac{\mathcal{N}_2\langle 1234\rangle^2 d\mu}{\langle AB_1AB_2\rangle\prod_{i=1}^{2}\langle AB_i12\rangle\langle AB_i23\rangle\langle AB_i34\rangle\langle AB_i14\rangle}$$

3-Loop: Similar structure, but more complicated...

We can represent the negative geometry expansion via sums over connected graphs:

$$\widetilde{\Omega}_L = \sum_G (-1)^{E(G)}$$ 

Ladder and Tree Resummation

[Arkani-Hamed, Henn, Trnka]

Results:

$$\mathcal{F}^{\text{ladder}}(g,z) \xrightarrow{g\gg 1} 0\,, \quad \Gamma^{\text{ladder}}(g) \xrightarrow{g\gg 1} 4\sqrt{2}g - 4\frac{\log 2}{\pi} + \frac{4}{\pi}e^{-2\sqrt{2}\pi g} + \cdots$$

$$\mathcal{F}^{\text{tree}}(g,z) \xrightarrow{g\gg 1} \frac{-z}{(1+z)^2}\,, \quad \Gamma^{\text{tree}}(g) \xrightarrow{g\gg 1} \frac{8}{\pi}g + \frac{1}{\pi}\frac{1}{g} + \cdots$$

$$\mathcal{F}^{\text{known}}(g,z) \xrightarrow{g\gg 1} \mathcal{O}(g)\,, \quad \Gamma^{\text{known}}(g) \xrightarrow{g\gg 1} 2g - \frac{3\log 2}{2\pi} + \cdots$$

How does this compare to known perturbative results from integrability?

$r_{\text{ladder/cusp}} =$

g^2	g^4	g^6	g^8	g^{10}	g^{12}	g^{14}	$\cdots$
1	1	0.73	0.44	0.25	0.14	0.07	$\cdots$

$r_{\text{cusp/tree}} =$

g^2	g^4	g^6	g^8	g^{10}	g^{12}	g^{14}	$\cdots$
1	1	0.92	0.83	0.74	0.63	0.53	$\cdots$

Does 1-loop in loop space bring us closer? Our aim!

Fixing Integrands

Denominator: Fixed by pole structure

$$\frac{1}{\prod_{i=1}^{L}\prod_{a=1}^{4}\langle AB_i a a+1\rangle} \times \frac{1}{\prod_{ij}\langle AB_i AB_j\rangle}$$

Fixing Integrands

Denominator: Fixed by pole structure

$$\frac{1}{\prod_{i=1}^{L}\prod_{a=1}^{4}\langle AB_i aa+1\rangle} \times \frac{1}{\prod_{ij}\langle AB_i AB_j\rangle}$$

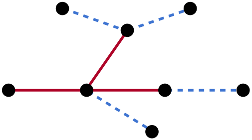
Numerator: Must vanish on unphysical cuts:

1. Cuts that violate the positivity or negativity of the links or vertices: Involve one link i.e. two vertices at a time.
2. Cuts that expose a double pole: Involve all vertices in a cycle.

Second cut is irrelevant in the tree case and as a result, the tree numerator can be built link by link.

Tree Result

$$\Omega_G^{\text{tree}} =$$



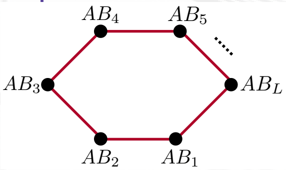
$$= \frac{d\mu\, \mathcal{N}_G}{\prod_i D_i \cdot \prod_{\text{links}} \langle AB_i AB_j \rangle}$$

$$\mathcal{N}_G = \langle 1234 \rangle^{L+1} \times \prod_{\text{pos links}} N_{ij}^{(+)} \times \prod_{\text{neg links}} N_{ij}^{(-)}$$

$$N_{ij}^{(+)} \equiv \langle AB_i 12 \rangle \langle AB_j 34 \rangle + \text{cyc}$$

$$N_{ij}^{(-)} \equiv \langle AB_i 13 \rangle \langle AB_j 24 \rangle + \langle AB_i 24 \rangle \langle AB_j 13 \rangle$$

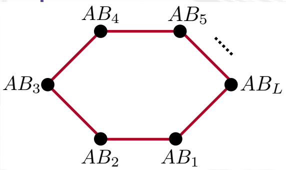
1-Loop Result



$$\mathcal{N}_G = \prod_{i=1}^{L} N_{i,i+1}^{(-)} + (-1)^L R_L^{1-\text{loop}}$$

$$R_L^{1-\text{loop}} = \Bigg\{ (2^L - 4) N_1^{(a)} N_2^{(a)} \cdots N_L^{(a)} - \sum_{p \in \text{even}} \sum_{i} \left\{ N_{i_1 i_2 \ldots i_p}^{(a)} N_{i_1 i_2 \ldots i_p}^{(c)} N_{i_p+1}^{(a)} \cdots N_{i_L}^{(a)} \right\} \Bigg\} + (a \to b)$$

1-Loop Result



$$\mathcal{N}_G = \prod_{i=1}^{L} N_{i,i+1}^{(-)} + (-1)^L R_L^{1-\text{loop}}$$

$$R_L^{1-\text{loop}} = \Big\{(2^L-4) N_1^{(a)} N_2^{(a)} \cdots N_L^{(a)} - \sum_{p\in\text{even}} \sum_i \{ N_{i_1 i_2 \ldots i_p}^{(a)} N_{i_1 i_2 \ldots i_p}^{(c)} N_{i_p+1}^{(a)} \cdots N_{i_L}^{(a)} \} \Big\} + (a \to b)$$

$$N_{12\ldots p}^{(a)} = \prod_{j\in\text{odd}} \langle AB_j 12\rangle \prod_{k\in\text{even}} \langle AB_k 34\rangle + \prod_{j\in\text{odd}} \langle AB_j 34\rangle \prod_{k\in\text{even}} \langle AB_k 12\rangle$$

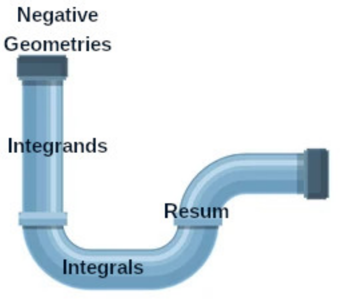
$$N_{12\ldots p}^{(b)} = \prod_{j\in\text{odd}} \langle AB_j 23\rangle \prod_{k\in\text{even}} \langle AB_k 14\rangle + \prod_{j\in\text{odd}} \langle AB_j 14\rangle \prod_{k\in\text{even}} \langle AB_k 23\rangle$$

$$N_{12\ldots p}^{(c)} = \prod_{j\in\text{odd}} \langle AB_j 13\rangle \prod_{k\in\text{even}} \langle AB_k 24\rangle + \prod_{j\in\text{odd}} \langle AB_j 24\rangle \prod_{k\in\text{even}} \langle AB_k 13\rangle$$

[Brown, Oktem, SP, Trnka]

Moving Forward

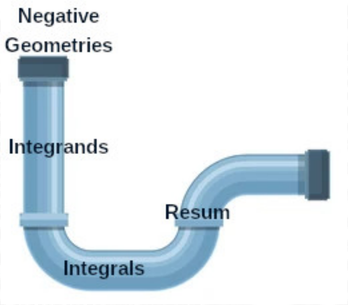
Ideally, we have the pipeline:



$\mathcal{F}(g, z)$
$\Gamma_{\text{cusp}}(g)$

Moving Forward

Ideally, we have the pipeline:



$\mathcal{F}(g,z)$
$\Gamma_{\text{cusp}}(g)$

But for generic cycles, the integrated result is not known.

The 3-loop and 4-loop cycle (i.e. the triangle and box) are known. We can use differential equations to find triangle/box + branches.

And in this case, resummation is possible.

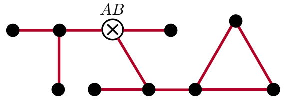
Triangles and Friends

We can use differential equations and the following conditions:

1. Result can be written in terms of harmonic polylogarithms (HPLs)
2. $\mathcal{F}(z \to -1) = 0$
3. $\mathcal{F}(\frac{1}{z}) = \mathcal{F}(z)$

The first can be seen from Landau analysis. The last two are consequences of the structure of the integrand.

This gives us all integrals of the type:



[Brown, Oktem, SP, Trnka]

Integrated Results: Single Branch

$$
\begin{aligned}
I_{3,1}(z) = {}& 8H_{0,0,0,0,0,0}(z) + 8H_{0,0,-1,0,0,0}(z) - 16H_{0,0,-1,-1,0,0}(z) \\
& + 8H_{0,0,-2,0,0}(z) - 8\zeta_3(2H_{0,0,-1}(z) - H_{0,0,0}(z)) \\
& + 4\pi^2(H_{0,0,-1,0}(z) - H_{0,0,-1,-1}(z) + H_{0,0,-2}(z)) + 13\zeta_4 \log^2(z) \\
& + C_{3,1}\log(z) + D_{3,1}\,,
\end{aligned}
$$

To fix the constants that, we impose the boundary conditions

$$
\begin{aligned}
C_{3,1} &= -16\zeta_{3,2} + 12\pi^2\zeta_3 - 80\zeta_5 \\
D_{3,1} &= 16\zeta_{3,3} - 16\zeta_3^2 + \frac{289}{2}\zeta_6.
\end{aligned}
$$

Integrated Results: n-Branch Triangle

$$I_{3,n}(z) = (-1)^n \Big[8H_{\vec{0}_{2n},0,0,0,0}(z) + 8H_{\vec{0}_{2n},-1,0,0,0}(z) - 16H_{\vec{0}_{2n},-1,-1,0,0}(z) + 8H_{\vec{0}_{2n},0,-1,0,0}(z)$$
$$- 16\zeta_3\, H_{\vec{0}_{2n},-1}(z) + 4\pi^2 \Big(H_{\vec{0}_{2n},-1,0}(z) - 2H_{\vec{0}_{2n},-1,-1}(z) + H_{\vec{0}_{2n},-2}(z) \Big)$$
$$+ 26\zeta_4 \frac{1}{(2n)!} \log^{2n}(z) + 8\zeta_3 \frac{1}{(2n+1)!} \log^{2n+1}(z)$$
$$+ \sum_{k=1}^{n} (-1)^k \left(\frac{C_k}{[2(n-k)+1]!} \log z^{2(n-k)+1} + \frac{D_k}{[2(n-k)]!} \log z^{2(n-k)} \right) \Big].$$

where the constants are fixed recursively using the boundary conditions.

Note: Only harmonic polylogarithms appear here i.e. the only branch points/singularities are at $z = 0, 1, -1$.

[Dixon, Oktem, SP, Trnka, Zhang, Xu] (to appear)

Extracting Γ_{cusp}

We have another integral to perform,

$$g\frac{\partial}{\partial g}\Gamma_{\text{cusp}} = \oint F(z)dz$$

Integrating gives,

	g^2	g^4	g^6	g^8	g^{10}	g^{12}	g^{14}
$\Gamma_{\text{tree}}/\Gamma_{\text{true}}$	1	1	1.091	1.198	1.353	1.579	1.893
$\Gamma_{\triangle,n}/\Gamma_{\text{true}}$	0	0	-0.091	-0.053	-0.059	-0.074	-0.097
$\Gamma_{\text{tree},\triangle,n}/\Gamma_{\text{true}}$	1	1	1	1.145	1.294	1.505	1.796

[Dixon, Oktem, SP, Trnka, Zhang, Xu] (to appear)

Γ_{cusp} for n-branch Triangle

$$\mathcal{I}_n = \beta_n + (-1)^n \Bigg[16\,\zeta_{2n+2,2} - 32 \sum_{k=1}^{n} k\,\zeta_{2k+1}\,\zeta_{2(n-k)+3} + \sum_{j=0}^{n} \xi_{j+1}\,\pi^{2j} \Big(-\zeta_{3,2(n-j)+1} + [2(n-j)+1]\zeta_{2(n-j)+2,2} + (n-j+1)[2(n-j)+1]\zeta_{2(n-j)+3,1} \Big) \Bigg],$$

where β_n is purely made of even zeta values, i.e. powers of π.

It is clear that this result has uniform transcendentality of $2n+4$ i.e. $2L$.

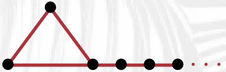
Similar formula available for box + n-branch.

[Dixon, Oktem, SP, Trnka, Zhang, Xu] (to appear)

Resumming Γ_{cusp}

Since we know the resummation of the ladder, we can focus on the cusp anomalous dimension:

$$\oint F_{\triangle}(z)F_{\text{ladder}}(z) = g\frac{\partial}{\partial g}\Gamma_{\text{cusp}}\Bigg[$$



$$\Bigg]$$

$$\overset{g\gg 1}{\sim} \frac{11}{45}\pi^3\sqrt{2}g^5 - \left(8\gamma + \frac{\pi^2}{3}\right)g^4 + \frac{\sqrt{2}\left(4\zeta_3 + \pi^2 - 3\gamma - 2\right)}{\pi}g^3 + \cdots$$

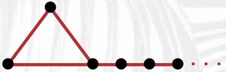
$$\overset{g\gg 1}{\sim} 10.71g^5 - 7.98g^4 + 4.86g^3 - 2.07g^2 + 0.47g$$

Does this mean that the expansion will not work at strong coupling? Maybe.

Resumming Γ_{cusp}

Since we know the resummation of the ladder, we can focus on the cusp anomalous dimension:

$$\oint F_{\triangle}(z)F_{\text{ladder}}(z) = g\frac{\partial}{\partial g}\Gamma_{\text{cusp}}\Bigg[$$



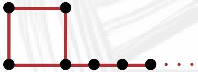
$$\Bigg]$$

$$\overset{g\gg 1}{\sim} \frac{11}{45}\pi^3\sqrt{2}g^5 - \left(8\gamma + \frac{\pi^2}{3}\right)g^4 + \frac{\sqrt{2}\,(4\zeta_3 + \pi^2 - 3\gamma - 2)}{\pi}g^3 + \cdots$$

$$\overset{g\gg 1}{\sim} 10.71g^5 - 7.98g^4 + 4.86g^3 - 2.07g^2 + 0.47g$$

Does this mean that the expansion will not work at strong coupling? Maybe. Yet the contribution from the box is

$$\oint F_{\square}(z)F_{\text{ladder}}(z) = g\frac{\partial}{\partial g}\Gamma_{\text{cusp}}\Bigg[$$

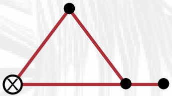


$$\Bigg]$$

$$\overset{g\gg 1}{\sim} 1.88g^7 - 2.13g^6 - 0.04g^5 + 0.83g^4 - 0.73g^3 + 0.36g^2 - 0.08g$$

The 1-cylce contributions can resum!

Other Geometries



$\mathrm{Symbol}(I_{3,2}) =$

$$
\begin{aligned}
&12\,\big(8\,\mathsf{SB}(z,z,z,z,b,b) + 2\,\mathsf{SB}(z,z,z,b,z,b) + 2\,\mathsf{SB}(z,z,z,b,b,z)\\
&+28\,\mathsf{SB}(z,z,z,z,z,z) + 8\,\mathsf{SB}(z,z,z,z,z,1{+}z) + 16\,\mathsf{SB}(z,z,z,z,1{+}z,z)\\
&-8\,\mathsf{SB}(z,z,z,z,1{+}z,1{+}z) + 18\,\mathsf{SB}(z,z,z,1{+}z,z,z) - 2\,\mathsf{SB}(z,z,z,1{+}z,z,1{+}z)\\
&-2\,\mathsf{SB}(z,z,z,1{+}z,1{+}z,z) + 30\,\mathsf{SB}(z,z,1{+}z,z,z,z) - 6\,\mathsf{SB}(z,z,1{+}z,z,z,1{+}z)\\
&-22\,\mathsf{SB}(z,z,1{+}z,z,1{+}z,z) - 32\,\mathsf{SB}(z,z,1{+}z,1{+}z,z,z)\big)\,,
\end{aligned}
$$

where

$$b = \frac{1 + i\sqrt{z}}{1 - i\sqrt{z}}\,, \qquad z = s/t\,.$$

The additional letter b shows up at individual negative geometry (starting from 4th entry), but drops out in the $F^{(3)}$ after sum.

Features of the Result

- Γ_{cusp} for tree graphs at L-loop has transcendentality $2L$ with single zeta values, but no MZVs.
- As the number of cycles increases, the depth of the MZVs also increase e.g. 1-cycle has MZVs of depth 2.
- Integrated 1-cycle results are made of HPL's rather than GPL's (spurious letters that appear at higher-cycle cancel).

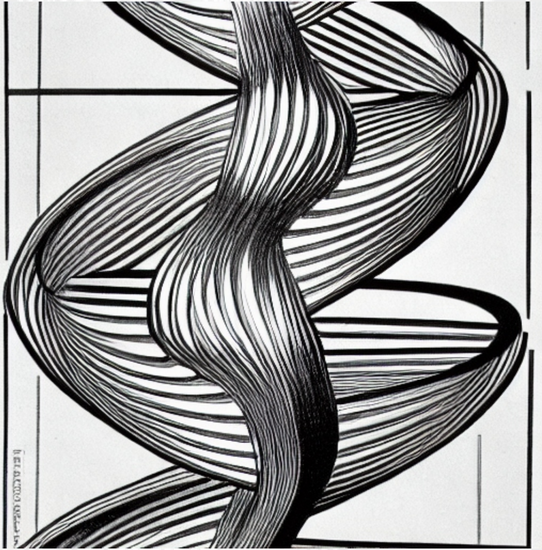
[SP, Skowronek, Spradlin, Volovich] (WIP)

- Higher the weight of the starting HPL, higher the g power of the resummed contribution.

Future Directions

- Can the other 1- or higher-cycle integrands be integrated?
- Can all 1-cycle contributions be resummed?
- If yes, will that give us a good approximation to the cusp anomalous dimension?
- What other quantities can we use similar approximations for?
- Can geometric Landau analysis prove that the only singularities and branch points will occur at $z = 0, 1, -1$?

$$\vdots$$



ଧନ୍ୟବାଦ