Building Blocks of Gravity Amplitudes

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Based on arXiv:2309.05710 with J Trnka

Outline

The more we study gravity, the more interesting properties about its amplitudes surface.

Is there a universal framework (possibly geometric) or construction of gravity that can explain its properties?

In planar $\mathcal{N} = 4$ SYM, the amplituhedron for e.g. plays this role.
New framework may allow new calculations.

What expression of the gravity amplitude could evince its properties?

Properties of Gravity

- BCJ: Gravity can be recognized as the double copy of Yang-Mills.
- CHY: Tree amplitudes are given by an integral over punctures on a sphere.
- UV behaviour:
  - Tree amplitudes fall off quicker than YM for large BCFW shift
  - Cuts of loop amplitudes in $\mathcal{N} = 8$ SUGRA also fall off quicker than expected
  - Loop amplitudes of $\mathcal{N} = 8$ SUGRA have fewer divergences than counting suggests
- Surprisingly simple MHV and NMHV formulae

Could these properties be consequences of a geometric construction?

Why do we think there might be this picture?

Planar $\mathcal{N} = 4$ SYM

Amplituhedron is a positive geometry. Amplitudes are given by a form on this space with logarithmic singularities on the boundaries.

It evinces various properties:

- Unitarity and locality
- Dual conformal (and Yangian) invariance
- Good large BCFW scaling
- BCJ? (Associahedron, causal diamonds...)

[Volovich, Spradlin]

Is there such a geometric picture that can be extended to gravity?

Difficulties

- Singularities in gravity are non-logarithmic,

  e.g. For YM MHV amplitudes i.e. Parke-Taylor, $\langle ij\rangle \to 0 \sim \frac{1}{\langle ij\rangle}$.

  For gravity MHV amplitudes, $\langle ij\rangle \to 0 \sim \frac{[ij]}{\langle ij\rangle}$.

- Amplitudes in gravity are permutational symmetric i.e. there is no ordering of the external states and so positive geometries defined in terms of momentum twistors cannot be used.
  There are possible ways around this.

  [Damgaard, Ferro, Lukowski, Parisi][Trnka, SP]

- Gravity has a dimension-full coupling. Already at the level of on-shell diagrams, this leads to kinematical dressings and makes things more complicated.

  [Hermann, Trnka][Heslop, Lipstien]

What can we do?

These make a direct approach difficult, so instead we will take the historic approach.

The amplituhedron resulted from a detailed study of the building blocks of $\mathcal{N} = 4$ SYM i.e. the BCFW terms or R-invariants.

There are some “nice” expressions for graviton amplitudes as well, each manifests a particular property:

- CHY manifests KLT
- Hodges formula for MHV manifests permutation symmetry

  $\vdots$

What are the right variables? What are good building blocks of gravity?

Strategy

1. Study BCFW terms in gravity

2. Exploit enhanced UV behaviour of BCFW-shifted gravity to express the amplitude as a sum of 1-loop LS

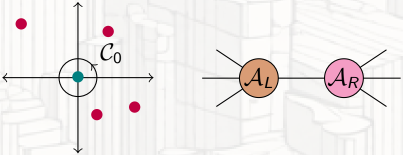
Strategy

1. Study BCFW terms in gravity
2. Exploit enhanced UV behaviour of BCFW-shifted gravity to express the amplitude as a sum of 1-loop LS !!!
2. Use knowledge of zeros of gravity to construct building blocks of gravity analogous to YM R-invariants
3. Check if the geometric properties extend to these "G-invariants"

BCFW Shifts

and corresponding global residue theorems allow us to construct YM and gravity:

$$
\begin{aligned}
\hat{\tilde{\lambda}}_1 =& |\hat{1}\,] = \tilde{\lambda}_1 + z\tilde{\lambda}_n = |1\,] + z\,|n\,] \\
\hat{\lambda}_n =& |\hat{n}\,\rangle = \lambda_n - z\lambda_1 = |n\,\rangle - z\,|1\,\rangle \\
\hat{\eta}_{1A} =& \eta_{1A} + z\eta_{nA}
\end{aligned}
$$



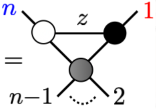
The true physical amplitude must be evaluated on unshifted kinematics:

$$
\mathcal{A}_n(0) = \sum_I \operatorname*{Res}_{z=z_I} \frac{\mathcal{A}_L(z)\mathcal{A}_R(z)}{P_I^2}.
$$

Recast as On-Shell Diagrams

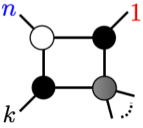
The BCFW recursion relations can be rewritten instead in terms of on-shell functions, as a residue theorem on a triple cut function,

$$\frac{s_{n1}}{z}\mathcal{A}_n^{\mathrm{GR}}(z) =$$

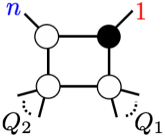


The residue theorem for $k=1$ (NMHV) leads to

$$\mathcal{A}_{n,1}^{\mathrm{GR}} = \frac{1}{s_{n1}}\sum_{k=2}^{n-1}$$



$$+\ \frac{1}{s_{n1}}\sum_{|Q_2|\geq 2}$$



[Britto, Cachazo, Feng]

1-Loop Leading Singularities Emerge

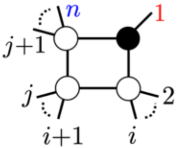
BCFW involves using the lower-point N^kMHV amplitude to build the higher-point N^kMHV amplitude i.e. $\mathcal{A}_{n-1,k} \Rightarrow \mathcal{A}_{n,k}$.

Can we instead recurse in k? i.e. $\mathcal{A}_{n,k-1} \Rightarrow \mathcal{A}_{n,k}$

In SYM, this allows us to write N^kMHV amplitudes in terms of (anti-)MHV vertices, which are just k-loop leading singularities.

For NMHV, 1-loop leading singularities i.e. R-invariants

$$\mathcal{A}_{n,1}^{\mathrm{YM}} = \sum_{j=i+2}^{n-1} \sum_{i=2}^{n-3}$$



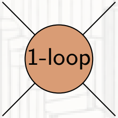
which are dual conformal invariant.

Recall R-Invariants

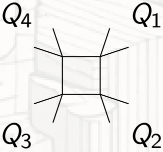
In YM, building blocks are R-invariants:

$$R[n(j-1)j(k-1)k] = \frac{\langle (j-1)j\rangle^4 \langle (k-1)k\rangle^4}{\langle n(j-1)j(k-1)\rangle \langle (j-1)j(k-1)k\rangle \langle j(k-1)kn\rangle \langle (k-1)kn(j-1)\rangle \langle kn(j-1)j\rangle}$$

which are 1-loop leading singularities:



$= [\text{R-inv}] \times$



In fact, all N^{k}MHV amplitudes in YM can be written as sums of products of R-invariants i.e. k-loop LS.

What is the analogous kinematic object in gravity?

Double Bonus Relations

To do the same in gravity, we need:

- Enhanced large z scaling
- Zeros: All graviton amplitudes satisfy collinear splitting theorems,

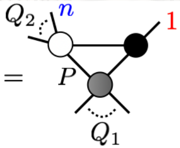
$$\mathcal{A}_{n,k} \xrightarrow{\langle ij\rangle \to 0} \frac{[ij]}{\langle ij\rangle}$$

In the case of the triple cut function, this allows us to write "double bonus" residue theorems,

$$\oint dz \mathcal{F}(z) = \oint dz(1+\alpha z)\mathcal{F}(z) = \oint dz \frac{(1+\alpha z+\beta z^2)}{z}\mathcal{F}(z) = 0$$

where we use a more general triple cut function,

$$\mathcal{F}(z) =$$



with

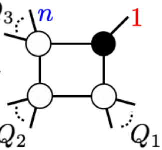
$$\lambda_P = (Q_1|1] + z\widetilde{\lambda}_1)$$

$$\widetilde{\lambda}_P = \frac{Q_1|1\rangle}{\langle 1|Q_1|1]}$$

Final Formula: NMHV Amplitudes

This gives us the final formula for NMHV amplitudes,

$$\mathcal{A}_{n,1}^{\mathrm{GR}} = \sum_{Q_1,Q_2} \frac{\langle 1|Q_1 Q_2|n\rangle}{\langle 1|Q_1 Q_2 Q_3|1]\langle 1n\rangle}$$



Allowing us to write a new expression for tree-level NMHV graviton amplitudes in terms of dressed one-loop leading singularities.

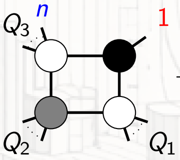
In SYM, all we needed was $\frac{1}{z}$ falloff as $z \to \infty$ and cyclicity.

These are the gravitational analogs of R-invariants i.e. G-invariants.

N^2MHV Amplitudes

Moving on to $k = 2$,

$$\mathcal{A}_{n,2}^{\mathrm{GR}} = \sum_{Q_1,Q_2} \frac{\langle P_3 n\rangle}{\langle 1n\rangle \mathcal{J}} \left(\quad \right)$$



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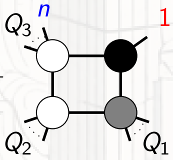
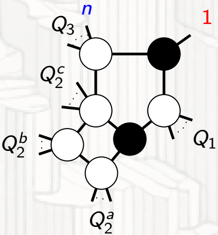


where each of the individual pieces are now 2-loop singularities. For example,

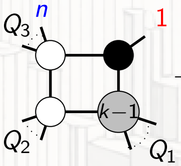
$$\frac{\langle P_3 n\rangle}{\langle 1n\rangle \mathcal{J}} \frac{\langle P_2|Q_2^a Q_2^b|P_3\rangle}{\langle P_2 P_3\rangle [P_2|Q_2^c Q_2^b Q_2^a|P_2\rangle}$$



General Formula

For general k, we get k-loop singularities with the same prefactors,

$$\mathcal{A}^{\mathrm{GR}}_{n,k} = \sum_{Q_1,Q_2} \frac{\langle P_3 n\rangle}{\langle 1 n\rangle \mathcal{J}} \Bigg($$



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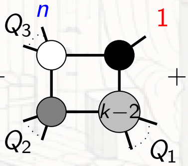
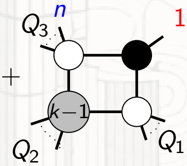


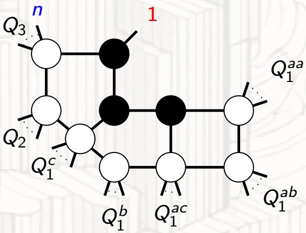
$+ \dots +$



$$\Bigg)$$

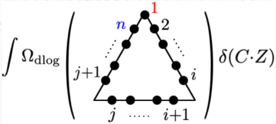
e.g. 3-loop singularities at N^3MHV,

$$\frac{\langle P_3 n\rangle}{\langle 1 n\rangle \mathcal{J}} \frac{\langle \tilde{P}_3 P_2\rangle}{\langle 1 P_2\rangle [P_1|Q_1^c Q_1^b Q_1^a|1\rangle} \frac{\langle 1|Q_1^{aa} Q_1^{ab}|\tilde{P}_2\rangle}{\langle 1 \tilde{P}_2\rangle [\tilde{P}_1|Q_1^{ac} Q_1^{ab} Q_1^{aa}|1\rangle}$$



G-invariants vs R-invariants

Dual formulation of $\mathcal{N}=4$ SYM BCFW terms as a dlog form on a convex space,



[Arkani-Hamed, Cachazo, Cheung, Kaplan, Bourjaily, Trnka, Goncharov, Postnikov...]

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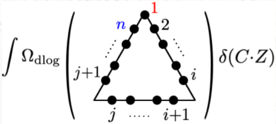


[Arkani-Hamed, Cachazo, Cheung, Kaplan, Bourjaily, Trnka, Goncharov, Postnikov...]

Can we write amplitudes in gravity as

$$\mathcal{A}_{\mathrm{GR}} = \int \Omega_{\mathrm{GR}}?$$

The first step would be to check whether our G-invariants share the geometric properties of R-invariants.

Geometry of R-Invariants

These objects satisfy two properties:

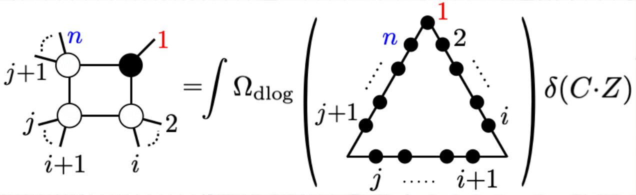
1. Shifting identity: Some of the R-invariants are equal
2. Six-term identity: Sum of the R-invariants is zero

This inspired one of the first polytope formulae for $\mathcal{N} = 4$ SYM NMHV amplitudes.

Will our building blocks also satisfy analogous properties?

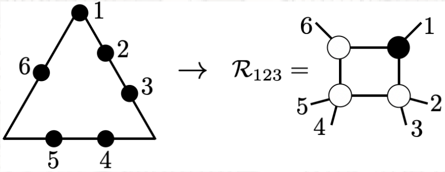
Geometric Property 1

For tree-level amplitudes,

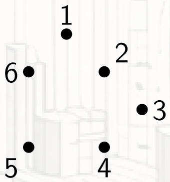


In YM, these configurations evaluate (integrate) to R-invariants,

$R[1,3,4,5,6] =$



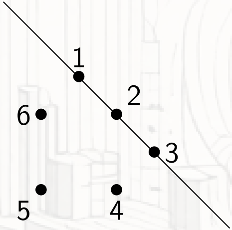
Geometric Relations



The invariant piece of geometric information is that $(123) = 0$.

Since *R*-invariants are "nice" objects every one associated to this configuration is equal.

Geometric Relations



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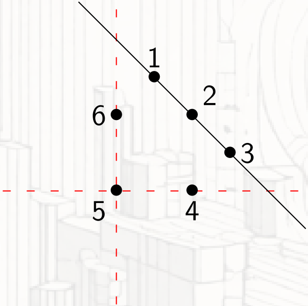

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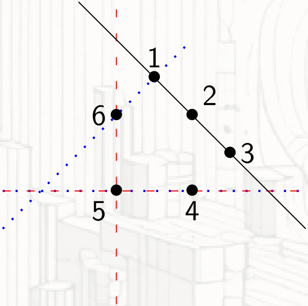

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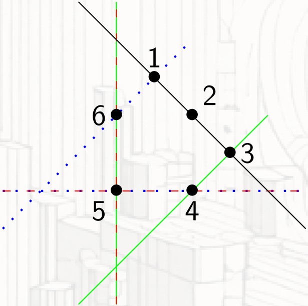

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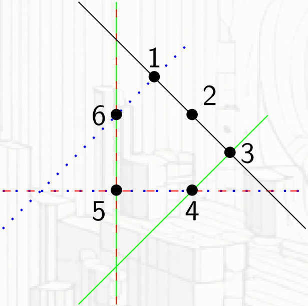

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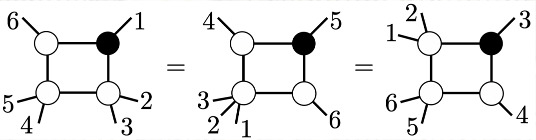
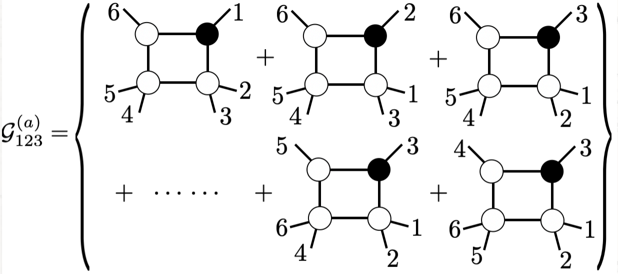
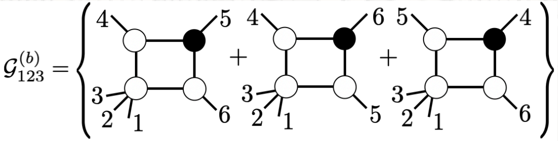


G-invariants Satisfy Property 1





$$\mathcal{G}_{123}^{(a)} = \mathcal{G}_{123}^{(b)} \equiv \mathcal{G}_{123}$$

Can be shown for any n via global residue theorems on triple cut functions.

Geometric Property 2: Six-Term Identity

Additionally YM R-invariants satisfy a nice identity,

$$\mathcal{R}_{123} + \mathcal{R}_{234} + \mathcal{R}_{345} + \mathcal{R}_{456} + \mathcal{R}_{561} + \mathcal{R}_{612} = 0$$

This is equivalent to the statement that shifting any two legs (ij) in BCFW gives the same result.

This statement is kinematically true, is a result of a residue theorem of a 1-form on $G(k,n)$.

[Arkani-Hamed, Cachazo, Cheung, Kaplan, Bourjaily, Trnka, Goncharov, Postnikov]

Can this too be true for our G-invariants?

Twenty-Term Identity

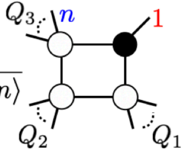
In fact, it is!

$$\sum_{S_6} \mathcal{G}_{abc} = 0$$

Remember that,

- We are strictly at 6-point so far.
- The G-invariants are exactly the 1-loop leading singularities we encountered in our formula but without the prefactors,

$$\mathcal{A}_{n,1}^{\mathrm{GR}} = \sum_{Q_1,Q_2} \frac{\langle 1|Q_1 Q_2|n\rangle}{\langle 1|Q_1 Q_2 Q_3|1]\langle 1n\rangle}$$



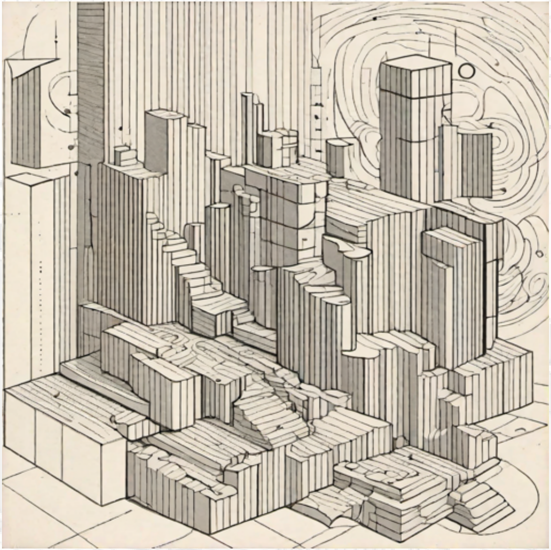
- Preliminary results show that we can rewrite the amplitude in terms of undressed G-invariants.

Next Questions

- What are the correct geometric objects in gravity? Is it dressed or undressed k-loop leading singularities?
- Why can we interpret Ω_{GR} as a line? In YM, it is very clear from the Grassmannian formula that Parke-Taylor is equivalent to the form on a line of ordered points in $\mathbb{P}^1$.
- Can we reformulate gravity as a "gravituhedron"?
- If so, does BCJ and UV behaviour follow?

Summary

- We presented a new expansion of tree-level gravity amplitudes in terms of dressed k-loop leading singularities i.e. gravitational R-invariants.
- The undressed objects have some geometric properties reminiscent of the configurations in $\mathbb{P}^{k+1}$ in YM.
- This opens up a new direction to pursue to find a geometric formulation of amplitudes in gravity.



Thank You