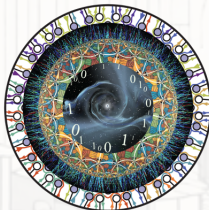


Building Blocks of Gravity Amplitudes



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QCD Meets Gravity 2023

Based on [arXiv:2309.05710](https://arxiv.org/abs/2309.05710) with J Trnka

Outline

The more we study gravity, the more interesting properties about its amplitudes surface.

Is there a universal framework (possibly geometric) or construction of gravity that can explain its properties?

In planar $\mathcal{N} = 4$ SYM, the amplituhedron for e.g. plays this role.
New framework may allow new calculations.

What expression of the gravity amplitude could evince its properties?

Properties of Gravity

- BCJ: Gravity can be recognized as the **double copy** of Yang-Mills.
- CHY: Tree amplitudes are given by an **integral** over punctures on a sphere.
- UV behaviour:
 - ▷ Tree amplitudes **fall off quicker** than YM for large BCFW shift
 - ▷ Cuts of loop amplitudes in $\mathcal{N} = 8$ SUGRA also **fall off quicker** than expected
 - ▷ Loop amplitudes of $\mathcal{N} = 8$ SUGRA have **fewer** divergences than counting suggests
- Surprisingly simple MHV and NMHV formulae

Could these properties be consequences of a **geometric construction**?

Why do we think there might be this picture?

Planar $\mathcal{N} = 4$ SYM

Amplituhedron is a positive geometry. Amplitudes are given by a form on this space with **logarithmic singularities** on the boundaries.

It evinces various properties:

- Unitarity and locality
- **Dual conformal** (and Yangian) invariance
- Good **large** BCFW scaling
- BCJ? (Associahedron, causal diamonds...)

[Volovich, Spradlin]

Is there such a geometric picture that can be extended to gravity?

Difficulties

- Singularities in gravity are **non-logarithmic**,
e.g. For YM MHV amplitudes i.e. Parke-Taylor, $\langle ij \rangle \rightarrow 0 \sim \frac{1}{\langle ij \rangle}$.
For gravity MHV amplitudes, $\langle ij \rangle \rightarrow 0 \sim \frac{[ij]}{\langle ij \rangle}$.
- Amplitudes in gravity are **permutational** symmetric i.e. there is **no ordering** of the external states and so positive geometries defined in terms of momentum twistors cannot be used.
There are possible ways around this.

[Damgaard, Ferro, Lukowski, Parisi][Trnka, SP]

- Gravity has a **dimension-full coupling**. Already at the level of on-shell diagrams, this leads to kinematical dressings and makes things more complicated.

[Hermann, Trnka][Heslop, Lipstien]

What can we do?

These make a direct approach **difficult**, so instead we will take the historic approach.

The amplituhedron resulted from a detailed study of the **building blocks of $\mathcal{N} = 4$ SYM** i.e. the BCFW terms or R-invariants.

There are some “nice” expressions for graviton amplitudes as well, each manifests a particular property:

- CHY manifests KLT
- Hodges formula for MHV manifests permutation symmetry
- $\vdots$

What are the **right variables**? What are good building blocks of gravity?

Strategy

1. Study BCFW terms in gravity
2. Exploit **enhanced UV behaviour** of BCFW-shifted gravity to express the amplitude as a sum of 1-loop LS

Strategy

1. Study BCFW terms in gravity
2. Exploit **enhanced UV behaviour** of BCFW-shifted gravity to express the amplitude as a sum of 1-loop LS !!!
2. Use knowledge of **zeros of gravity** to construct building blocks of gravity analogous to YM R-invariants
3. Check if the **geometric properties extend** to these “G-invariants”

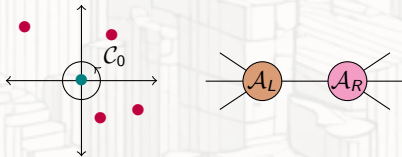
BCFW Shifts

and corresponding global residue theorems allow us to construct YM and gravity:

$$\hat{\lambda}_1 = |\hat{1}] = \tilde{\lambda}_1 + z\tilde{\lambda}_n = |1] + z|n]$$

$$\hat{\lambda}_n = |\hat{n}\rangle = \lambda_n - z\lambda_1 = |n\rangle - z|1\rangle$$

$$\hat{\eta}_{1A} = \eta_{1A} + z\eta_{nA}$$

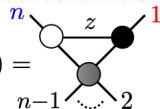


The true physical amplitude must be evaluated on unshifted kinematics:

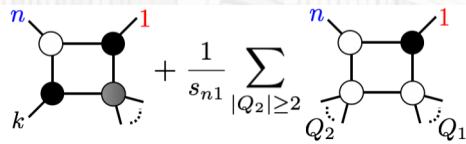
$$\mathcal{A}_n(0) = \sum_I \text{Res}_{z=z_I} \frac{\mathcal{A}_L(z)\mathcal{A}_R(z)}{P_I^2}.$$

Recast as On-Shell Diagrams

The BCFW recursion relations can be rewritten instead in terms of **on-shell functions**, as a residue theorem on a triple cut function,

$$\frac{s_{n1}}{z} \mathcal{A}_n^{\text{GR}}(z) =$$


The residue theorem for $k=1$ (NMHV) leads to

$$\mathcal{A}_{n,1}^{\text{GR}} = \frac{1}{s_{n1}} \sum_{k=2}^{n-1} \text{Diagram 1} + \frac{1}{s_{n1}} \sum_{|Q_2| \geq 2} \text{Diagram 2}$$


[Britto, Cachazo, Feng]

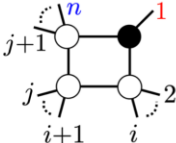
1-Loop Leading Singularities Emerge

BCFW involves using the **lower-point** $N^k\text{MHV}$ amplitude to build the **higher-point** $N^k\text{MHV}$ amplitude i.e. $\mathcal{A}_{n-1,k} \Rightarrow \mathcal{A}_{n,k}$.

Can we instead recurse in k ? i.e. $\mathcal{A}_{n,k-1} \Rightarrow \mathcal{A}_{n,k}$

In SYM, this allows us to write $N^k\text{MHV}$ amplitudes in terms of (anti-)MHV vertices, which are just **k -loop leading singularities**.

For NMHV, 1-loop leading singularities i.e. R-invariants

$$\mathcal{A}_{n,1}^{\text{YM}} = \sum_{j=i+2}^{n-1} \sum_{i=2}^{n-3} \text{Diagram}$$


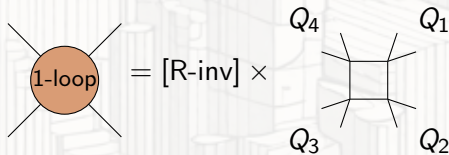
which are dual conformal invariant.

Recall R-Invariants

In YM, building blocks are **R-invariants**:

$$R[n(j-1)j(k-1)k] = \frac{\langle(j-1)j\rangle^4 \langle(k-1)k\rangle^4}{\langle n(j-1)j(k-1) \rangle \langle(j-1)j(k-1)k \rangle \langle j(k-1)kn \rangle \langle(k-1)kn(j-1) \rangle \langle kn(j-1)j \rangle}$$

which are **1-loop leading singularities**:



The diagram illustrates the relationship between a 1-loop singularity and R-invariants. On the left, a brown circle labeled "1-loop" has four external lines. This is equated to the product of an R-invariant, $[R\text{-inv}]$, and a box diagram. The box diagram consists of a central square with four external lines labeled Q_1 , Q_2 , Q_3 , and Q_4 at the corners.

In fact, all $N^k\text{MHV}$ amplitudes in YM can be written as **sums of products** of R-invariants i.e. k -loop LS.

What is the analogous kinematic object in gravity?

Double Bonus Relations

To do the same in gravity, we need:

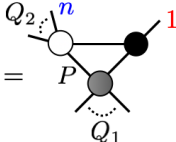
- Enhanced large z scaling
- Zeros: All graviton amplitudes satisfy **collinear splitting** theorems,

$$\mathcal{A}_{n,k} \xrightarrow{\langle ij \rangle \rightarrow 0} \frac{[ij]}{\langle ij \rangle}$$

In the case of the triple cut function, this allows us to write “double bonus” residue theorems,

$$\oint dz \mathcal{F}(z) = \oint dz (1 + \textcolor{violet}{\alpha} z) \mathcal{F}(z) = \oint dz \frac{(1 + \alpha z + \textcolor{violet}{\beta} z^2)}{\textcolor{violet}{z}} \mathcal{F}(z) = 0$$

where we use a more general triple cut function,

$\mathcal{F}(z) =$


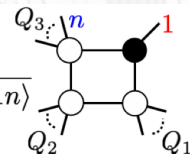
with

$$\lambda_P = (Q_1|1] + z\tilde{\lambda}_1)$$

$$\tilde{\lambda}_P = \frac{Q_1|1]}{\langle 1|Q_1|1]}$$

Final Formula: NMHV Amplitudes

This gives us the **final formula** for NMHV amplitudes,

$$\mathcal{A}_{n,1}^{\text{GR}} = \sum_{Q_1, Q_2} \frac{\langle 1|Q_1 Q_2|n \rangle}{\langle 1|Q_1 Q_2 Q_3|1 \rangle \langle 1n \rangle}$$


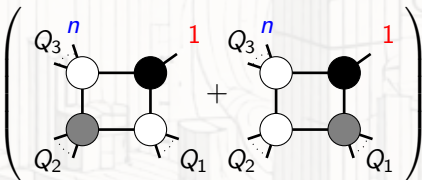
Allowing us to write a new expression for tree-level NMHV graviton amplitudes in terms of dressed one-loop leading singularities.

In SYM, all we needed was $\frac{1}{z}$ falloff as $z \rightarrow \infty$ and cyclicity.

These are the gravitational analogs of R -invariants i.e. **G-invariants**.

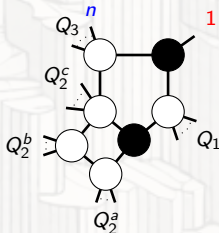
N²MHV Amplitudes

Moving on to $k = 2$,

$$\mathcal{A}_{n,2}^{\text{GR}} = \sum_{Q_1, Q_2} \frac{\langle P_3 n \rangle}{\langle 1 n \rangle \mathcal{I}} \left(\begin{array}{c} \text{Diagram 1} + \text{Diagram 2} \end{array} \right)$$


where each of the individual pieces are now 2-loop singularities. For example,

$$\frac{\langle P_3 n \rangle}{\langle 1 n \rangle \mathcal{I}} \frac{\langle P_2 | Q_2^a Q_2^b | P_3 \rangle}{\langle P_2 P_3 \rangle [P_2 | Q_2^c Q_2^b Q_2^a | P_2]}$$



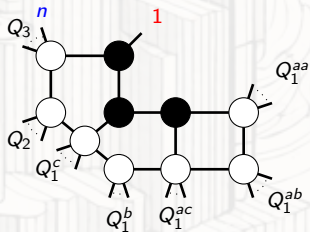
General Formula

For general k , we get k -loop singularities with the same prefactors,

$$\mathcal{A}_{n,k}^{\text{GR}} = \sum_{Q_1, Q_2} \frac{\langle P_3 n \rangle}{\langle 1 n \rangle \mathcal{J}} \left(\begin{array}{c} \text{Diagram 1} + \text{Diagram 2} + \dots + \text{Diagram } k \end{array} \right)$$

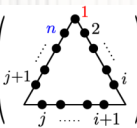
e.g. 3-loop singularities at $N^3\text{MHV}$,

$$\frac{\langle P_3 n \rangle}{\langle 1 n \rangle \mathcal{J}} \frac{\langle \tilde{P}_3 P_2 \rangle}{\langle 1 P_2 \rangle [P_1 | Q_1^c Q_1^b Q_1^a | 1]} \frac{\langle 1 | Q_1^{aa} Q_1^{ab} | \tilde{P}_2 \rangle}{\langle 1 \tilde{P}_2 \rangle [\tilde{P}_1 | Q_1^{ac} Q_1^{ab} Q_1^{aa} | 1]}$$



G-invariants vs R-invariants

Dual formulation of $\mathcal{N} = 4$ SYM BCFW terms as a dlog form on a convex space,

$$\int \Omega_{\text{dlog}} \left(\text{triangle diagram} \right) \delta(C \cdot Z)$$


[Arkani-Hamed, Cachazo, Cheung, Kaplan, Bourjaily, Trnka, Goncharov, Postnikov...]

G-invariants vs R-invariants

Dual formulation of $\mathcal{N} = 4$ SYM BCFW terms as a dlog form on a convex space,

$$\int \Omega_{\text{dlog}} \left(\begin{array}{c} \text{Diagram of a triangle with vertices } j+1, i, j \text{ and internal points } 1, 2, \dots, n \end{array} \right) \delta(C \cdot Z)$$

[Arkani-Hamed, Cachazo, Cheung, Kaplan, Bourjaily, Trnka, Goncharov, Postnikov...]

Can we write amplitudes in gravity as

$$\mathcal{A}_{\text{GR}} = \int \Omega_{\text{GR}}?$$

The first step would be to check whether our G-invariants share the geometric properties of R-invariants.

Geometry of R-Invariants

These objects satisfy **two** properties:

1. Shifting identity: Some of the R-invariants are equal
2. Six-term identity: Sum of the R-invariants is zero

This inspired one of the first **polytope formulae** for $\mathcal{N} = 4$ SYM NMHV amplitudes.

Will our building blocks also satisfy analogous properties?

Geometric Property 1

For tree-level amplitudes,

$$\begin{array}{c}
 \text{Diagram 1: A square-like graph with four vertices. The top-left vertex is white and labeled $j+1$ with an incoming blue line n. The top-right vertex is black and labeled 1 with an incoming red line. The bottom-left vertex is white and labeled j with an incoming dotted line. The bottom-right vertex is white and labeled i with an incoming dotted line. The bottom edge has two additional vertices labeled $i+1$ and i with incoming dotted lines.} \\
 \\
 \text{Diagram 2: A triangular graph with vertices labeled $j+1$ (bottom-left), j (bottom-left-most), $i+1$ (bottom-right-most), i (bottom-right), and 1 (top) with an incoming red line. The left edge has n vertices, the right edge has 2 vertices, and the bottom edge has i vertices.}
 \end{array}
 = \int \Omega_{\text{dlog}} \left(\begin{array}{c} \text{Diagram 2} \end{array} \right) \delta(C \cdot Z)$$

In YM, these configurations evaluate (integrate) to R -invariants,

$$R[1, 3, 4, 5, 6] = \begin{array}{c} \text{Diagram 3: A triangle with vertices labeled 1 (top), 6 (left), and 3 (right). The bottom edge has two vertices labeled 5 and 4 from left to right.} \end{array} \rightarrow \mathcal{R}_{123} = \begin{array}{c} \text{Diagram 4: A square-like graph with four vertices. The top-left vertex is white and labeled 6. The top-right vertex is black and labeled 1. The bottom-left vertex is white and labeled 5 with an incoming line 4. The bottom-right vertex is white and labeled 2 with an incoming line 3.} \end{array}$$

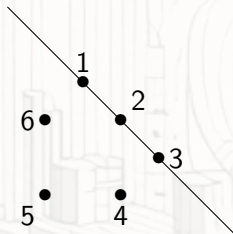
Geometric Relations



The invariant piece of geometric information is that $(123) = 0$.

Since R -invariants are “nice” objects every one associated to this configuration is **equal**.

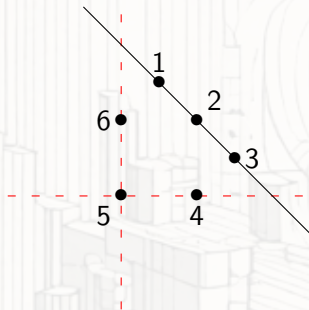
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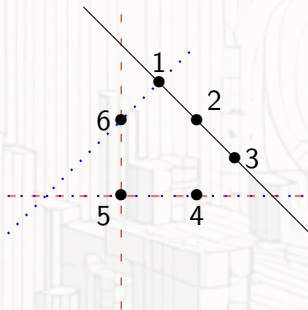
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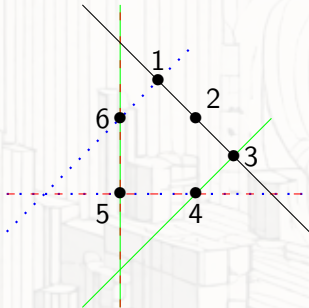
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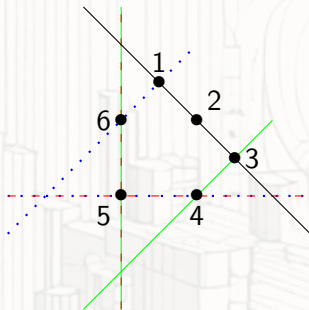
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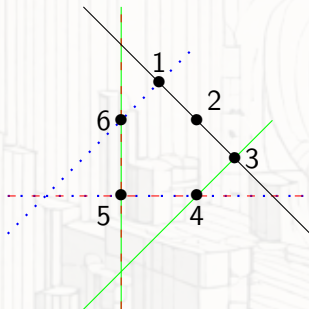
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The invariant piece of geometric information is that $(123) = 0$.

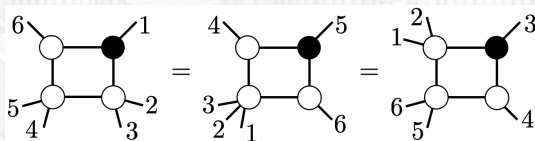
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Geometric Relations



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Since R -invariants are “nice” objects every one associated to this configuration is **equal**.



G-invariants Satisfy Property 1

$$\mathcal{G}_{123}^{(a)} = \left\{ \begin{array}{ccc} \begin{array}{c} \text{Diagram 1: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 6-1, 5-2, 4-3.} \\ \text{Diagram 2: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 6-2, 5-1, 4-3.} \\ \text{Diagram 3: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 6-3, 5-1, 4-2.} \end{array} & + & \begin{array}{c} \text{Diagram 4: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 5-3, 4-1, 6-2.} \\ \text{Diagram 5: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 4-3, 6-1, 5-2.} \end{array} & + & \dots & + & \begin{array}{c} \text{Diagram 6: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 4-3, 6-1, 5-2.} \\ \text{Diagram 7: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 4-3, 6-1, 5-2.} \end{array} \end{array} \right\}$$

$$\mathcal{G}_{123}^{(b)} = \left\{ \begin{array}{ccc} \begin{array}{c} \text{Diagram 8: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 4-5, 3-6, 2-1.} \\ \text{Diagram 9: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 4-6, 3-5, 2-1.} \\ \text{Diagram 10: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 5-4, 6-3, 1-2.} \end{array} & + & \begin{array}{c} \text{Diagram 11: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 4-6, 3-5, 2-1.} \\ \text{Diagram 12: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 5-4, 6-3, 1-2.} \end{array} & + & \dots & + & \begin{array}{c} \text{Diagram 13: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 5-4, 6-3, 1-2.} \\ \text{Diagram 14: Top-left white, top-right black, bottom-left white, bottom-right white. Edges: 5-4, 6-3, 1-2.} \end{array} \end{array} \right\}$$

$$\mathcal{G}_{123}^{(a)} = \mathcal{G}_{123}^{(b)} \equiv \mathcal{G}_{123}$$

Can be shown **for any** n via global residue theorems on triple cut functions.

Geometric Property 2: Six-Term Identity

Additionally YM R -invariants satisfy a nice identity,

$$\mathcal{R}_{123} + \mathcal{R}_{234} + \mathcal{R}_{345} + \mathcal{R}_{456} + \mathcal{R}_{561} + \mathcal{R}_{612} = 0$$

This is equivalent to the statement that shifting **any two legs (ij) in BCFW** gives the same result.

This statement is kinematically true, is a result of a residue theorem of a 1-form on $G(k, n)$.

[Arkani-Hamed, Cachazo, Cheung, Kaplan, Bourjaily, Trnka, Goncharov, Postnikov]

Can this too be true for our G -invariants?

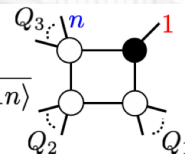
Twenty-Term Identity

In fact, **it is!**

$$\sum_{S_6} \mathcal{G}_{abc} = 0$$

Remember that,

- We are strictly at 6-point so far.
- The G-invariants are exactly the 1-loop leading singularities we encountered in our formula but **without the prefactors**,

$$\mathcal{A}_{n,1}^{\text{GR}} = \sum_{Q_1, Q_2} \frac{\langle 1|Q_1 Q_2|n \rangle}{\langle 1|Q_1 Q_2 Q_3|1 \rangle \langle 1n \rangle}$$


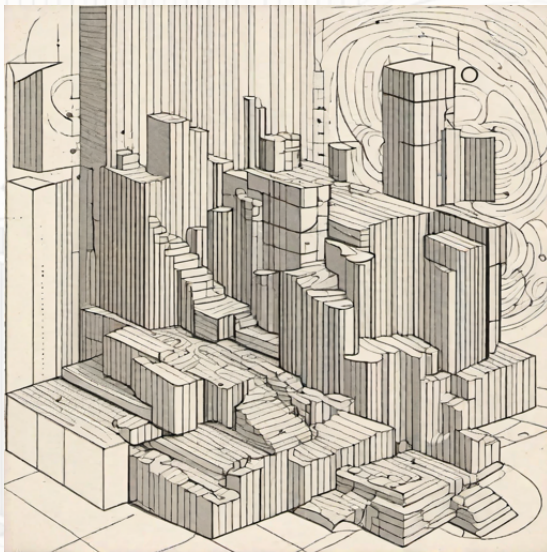
- Preliminary results show that we can rewrite the amplitude in terms of undressed G-invariants.

Next Questions

- What are the correct geometric objects in gravity? Is it **dressed or undressed k -loop leading singularities**?
- Why can we interpret Ω_{GR} as a **line**? In YM, it is very clear from the Grassmannian formula that Parke-Taylor is equivalent to the form on a line of ordered points in $\mathbb{P}^1$.
- Can we reformulate gravity as a “gravituhedron”?
- If so, does BCJ and UV behaviour follow?

Summary

- We presented a new expansion of tree-level gravity amplitudes in terms of dressed k -loop leading singularities i.e. **gravitational R -invariants**.
- The undressed objects have some geometric properties **reminiscent of the configurations in $\mathbb{P}^{k+1}$ in YM**.
- This opens up a new direction to pursue to find a geometric formulation of amplitudes in gravity.



Thank You