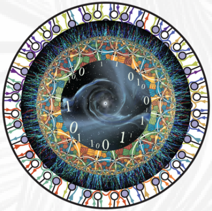
# Loops in a Loop Expansion



Shruti Paranjape

University of California, Davis
Center for Quantum Mathematics and Physics

Simons Center for Geometry and Physics
Workshop on Mathematical Aspects of $\mathcal{N} = 4$ SYM Theory

Based on arXiv:2312.17736 with T Brown, U Oktem, J Trnka

# Friendly Faces


all-loop 4-point partial amplitude in planar $\mathcal{N} = 4$ SYM

# Loops in a Loop Expansion

1. Why are we expanding?

# Loops in a Loop Expansion

1. Why are we expanding?
2. What are we expanding?

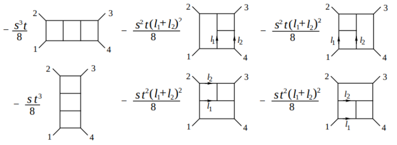
# Loops in a Loop Expansion

1. Why are we expanding?
2. What are we expanding?
3. How are we expanding?

# Why are we expanding?

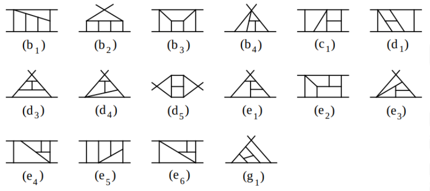
Can we resum perturbative results to calculate quantities at strong coupling?

$$\mathcal{A}_L = \int \mathcal{I}_L$$

e.g. $\mathcal{I}_3 = \Bigg\{$  $\Bigg\}$



[Bern, Dixon, Smirnov]

$\mathcal{I}_4 = \Bigg\{$  $\Bigg\}$

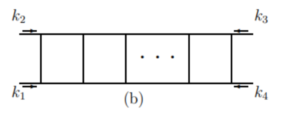


[Bern, Czakon, Dixon, Kosower, Smirnov]

# Resummation

Two examples of subsets of diagrams being resummed:

1. All ladders in $\phi^3$

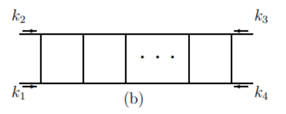



[Broadhurst, Davydychev]

# Resummation

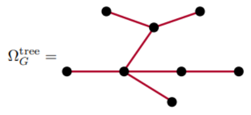
Two examples of subsets of diagrams being resummed:

1. All ladders in $\phi^3$




[Broadhurst, Davydychev]

2. All "trees in loop space" in $\mathcal{N} = 4$ SYM




[Arkani-Hamed, Henn, Trnka]

Both of these are ad-hoc selection of graphs to resum with different results. Does the resummation capture the strong coupling behaviour of the quantity of interest?

# This Talk

Yes! ... in some examples.

Why did this subset capture strong coupling behaviour? Is this a good approximation? Can we go to the next order?

This talk: Loops in a loop expansion

# This Talk

Yes! ... in some examples.

Why did this subset capture strong coupling behaviour? Is this a good approximation? Can we go to the next order?

This talk: Loops in a loop expansion

1. Why are we expanding? ✓
2. What are we expanding?
3. How are we expanding?

# What are we expanding?

At $L$-loop, the amplitude has divergences, $\mathcal{M}_L \sim \frac{1}{\epsilon^{2L}}$.

Divergences were shown to have special structure $\Rightarrow$ BDS Ansatz:

non-iterative terms

$E_n^{(l)}(0) = 0$

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} g^l \left(f^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

for $n = 4, 5$

# What are we expanding?

At $L$-loop, the amplitude has divergences, $\mathcal{M}_L \sim \frac{1}{\epsilon^{2L}}$.

Divergences were shown to have special structure $\Rightarrow$ BDS Ansatz:

non-iterative terms

$E_n^{(l)}(0) = 0$

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} g^l \left(f^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

constant

for $n = 4, 5$

# What are we expanding?

At $L$-loop, the amplitude has divergences, $\mathcal{M}_L \sim \frac{1}{\epsilon^{2L}}$.

Divergences were shown to have special structure $\Rightarrow$ BDS Ansatz:

non-iterative terms

$E_n^{(l)}(0) = 0$

all-$\epsilon$ 1-loop amplitude

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} g^l \left( f^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

constant

for $n = 4, 5$

# What are we expanding?

At $L$-loop, the amplitude has divergences, $\mathcal{M}_L \sim \frac{1}{\epsilon^{2L}}$.

Divergences were shown to have special structure $\Rightarrow$ BDS Ansatz:

non-iterative terms

$E_n^{(l)}(0) = 0$

all-$\epsilon$ 1-loop amplitude

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} g^l \left(f^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$

constant

for $n = 4, 5$

# What are we expanding?

At $L$-loop, the amplitude has divergences, $\mathcal{M}_L \sim \frac{1}{\epsilon^{2L}}$.

Divergences were shown to have special structure $\Rightarrow$ BDS Ansatz:

non-iterative terms

$E_n^{(l)}(0) = 0$

all-$\epsilon$ 1-loop amplitude

$$\mathcal{M}_n = \exp\left[\sum_{l=1}^{\infty} g^l \left(f^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

$$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$$

constant

cusp anomalous dimension

for $n = 4, 5$

# IR Finite Quantities

BDS ansatz suggests we should look at

$$\log \mathcal{M}_4 = \frac{\Gamma_{\text{cusp}}}{\epsilon^2} + \mathcal{O}(\epsilon)$$

The divergence is milder. But is there an IR finite quantity we can compute instead?

# IR Finite Quantities

BDS ansatz suggests we should look at

$$\log \mathcal{M}_4 = \frac{\Gamma_{\text{cusp}}}{\epsilon^2} + \mathcal{O}(\epsilon)$$

The divergence is milder. But is there an IR finite quantity we can compute instead?

Consider the integrand of $\log \mathcal{M}_L = \int \tilde{\Omega}_L$.

Since each integration gives $\frac{1}{\epsilon^2}$, doing all but one integral gives an IR finite quantity that can only depend on

$$z = \frac{\langle AB12\rangle\langle AB34\rangle}{\langle AB14\rangle\langle AB23\rangle}$$

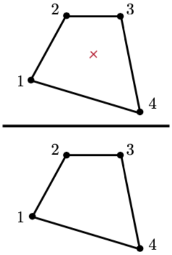
due to dual conformal invariance of the integrand.

# A Third Loop

We denote:

$$\mathcal{F}_L(z) = \int_{\text{all but one}} \tilde{\Omega}_L \quad \Rightarrow \quad \mathcal{F}(g,z) = \sum_{L=1}^{\infty} \mathcal{F}_L(z) g^{2L}$$

Wilson loop $\leftrightarrow$ amplitude duality tells us that this quantity is

$\mathcal{F}(g,z) =$ 


[Alday, Buchbinder, Tsytelin, Engelund, Roiban, Henn, Sikorowski, Korchemsky, Mistlberger, Chicherin...]

This integrand $\mathcal{F}(g,z)$ can be used to obtain $\Gamma_{\text{cusp}}$.

# Known Asymptotic Behaviour

Results from integrability, AdS/CFT but no strong coupling results from amplitudes: [Beisert, Eden, Staudacher]

$\mathcal{F}(g,z)$:

$$\to \begin{cases} -g^2 + g^4(\log^2 z + \pi^2) + \cdots & g \ll 1 \\ g\,\frac{z}{(z-1)^3}\,(2(1-z) + (z+1)\log z) + \cdots & g \gg 1 \end{cases}$$

We will not reproduce the behaviour of this function.

As we saw this is related to a particular ratio of Wilson loops.

[Alday, Henn, Sirkowski] [Henn, Korchemsky, Mistlberger]

# Known Asymptotic Behaviour

On the other hand:

$\Gamma(g)$:

$$\rightarrow \begin{cases} 4g^2 - 8\zeta_2 g^4 + \cdots & g \ll 1 \\ 2g - \frac{3\log 2}{2\pi} + \cdots & g \gg 1 \end{cases}$$

Perturbative regime results can be reproduced from amplitudes.

[Bern, Czakon, Dixon, Kosower, Smirnov]

Strong coupling regime results can be reproduced from minimal surface anchored to the boundary at a null polygon.

[Benna, Benvenuti, Klebanov, Scardicchio] [Basso,Korchemsky, Kotanski]

[Alday, Buchbinder, Tsytelin]

# This Talk

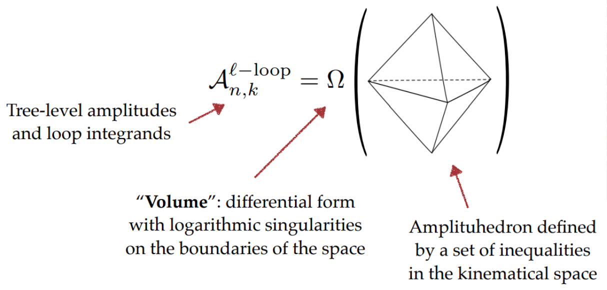
Can we get all-loop results beyond the “tree” approximation?

Can we capture $g \gg 1$ behaviour from the all-loop integrand given via the amplituhedron?

1. Why are we expanding? ✓
2. What are we expanding? ✓
3. How are we expanding?

# How are we expanding?

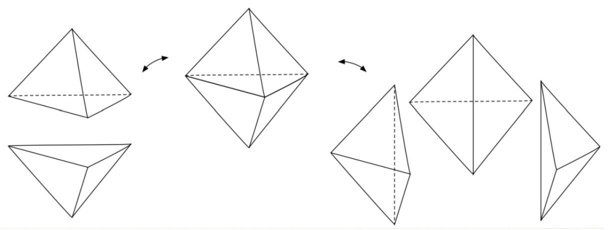
The amplituhedron is a polytope in twistor space whose volume gives tree amplitudes and loop integrands in $\mathcal{N} = 4$ SYM.




[Arkani-Hamed, Trnka]

# How are we expanding?

The amplituhedron is a polytope in twistor space whose volume gives tree amplitudes and loop integrands in $\mathcal{N} = 4$ SYM.

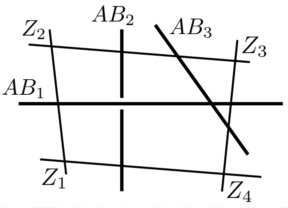


[Arkani-Hamed, Trnka]

In practice, calculating this volume form involves triangulation which is non-trivial. We take an algebraic approach.

# Four-Point Amplituhedron

At $L$-loop, introduce $L$ lines in momentum twistor space,




$Z_i \longrightarrow$ External momentum twistors
$AB_i \longrightarrow$ Loop lines

$Z_i$ satisfy $\langle 1234 \rangle > 0$, while $AB_i$ satisfy the following conditions:

$$\langle AB_i a a{+}1 \rangle > 0 \quad \langle AB_i 13 \rangle < 0 \quad \langle AB_i 24 \rangle < 0 \quad \langle AB_i AB_j \rangle > 0$$

We are looking for a canonical form on the space carved out by these inequalities that has logarithmic singularities on the boundaries.

# Form of the Form

The first set of inequalities reduce to

$$x_i, y_i, w_i, z_i > 0$$

where we use $Z_{A_i} = Z_1 + x_i Z_2 - y_i Z_4\,, \quad Z_{B_i} = Z_3 + w_i Z_2 + z_i Z_4\,.$

This part of the form is easy,

$$\Omega_L \sim \prod_{i=1}^{L} \frac{dx_i}{x_i}\frac{dy_i}{y_i}\frac{dz_i}{z_i}\frac{dw_i}{w_i}$$

The mutual conditions $\langle AB_i AB_j \rangle > 0$ are quadratic and harder to see. We can work simplex by simplex, e.g.

$$-(x_2-x_1)(w_2-w_1) - (y_2-y_1)(z_2-z_1) > 0 \Rightarrow w_2 > w_1 + \frac{(y_1-y_2)(z_1-z_2)}{(x_1-x_2)}$$

# Form of the Form

This in a particular simplex

$$x_1 > x_2, y_1 > y_2, z_1 > z_2, w_2 > w_1$$

$$\Omega_2^{(1)} \supset \frac{dw_2}{w_2 - w_1 - \frac{(y_1-y_2)(z_1-z_2)}{(x_1-x_2)}}.$$

Adding up the contributions,

$$\Omega_2 = \sum_{i=1}^{8} \Omega_2^{(i)}$$

This increases quickly in complexity.
Instead we will try to fix the form using cut conditions directly in $AB_i$ variables:

$$\Omega_L = \int \frac{\mathcal{N}_L}{\mathcal{D}_L} d\mu_L \quad d\mu_L = \prod_{i=1}^{L} \langle AB_i d^2 A_i \rangle \langle AB_i d^2 B_i \rangle$$

# Numerators

1-Loop:

$$\frac{\langle 1234\rangle^2 d\mu}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle} = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dz_1}{z_1}\frac{dw_1}{w_1}$$

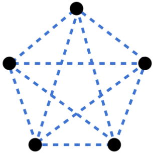
2-Loop: Now more complicated in $x, y, z, w$, involving mutual conditions.

$$\frac{\mathcal{N}_2\langle 1234\rangle^2 d\mu}{\langle AB_1AB_2\rangle\prod_{i=1}^{2}\langle AB_i12\rangle\langle AB_i23\rangle\langle AB_i34\rangle\langle AB_i14\rangle} \neq \prod_i \frac{dx_i}{x_i}\frac{dy_i}{y_i}\frac{dz_i}{z_i}\frac{dw_i}{w_i}$$

3-Loop: Even more complicated...

We can represent these conditions via a graph:

$\Omega_L =$



# Log Form

We would like $\log \mathcal{M}_L = \int \tilde{\Omega}_L$ where

$$
\begin{aligned}
\tilde{\Omega}_1 =& \Omega_1(AB) \\
\tilde{\Omega}_2 =& \Omega_2(AB, CD) - \Omega_1(AB)\Omega_1(CD) \\
\tilde{\Omega}_3 =& \Omega_3(AB, CD, EF) - \Omega_2(AB, CD)\Omega_1(EF) - \text{cyc} \\
& + 2\Omega_1(AB)\Omega_1(CD)\Omega_1(EF) \\
& \vdots
\end{aligned}
$$

Are these canonical forms on a different space?

# Log Form

We would like $\log \mathcal{M}_L = \int \tilde{\Omega}_L$ where

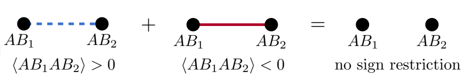
$$\begin{aligned}
\tilde{\Omega}_1 =& \Omega_1(AB) \\
\tilde{\Omega}_2 =& \Omega_2(AB, CD) - \Omega_1(AB)\Omega_1(CD) \\
\tilde{\Omega}_3 =& \Omega_3(AB, CD, EF) - \Omega_2(AB, CD)\Omega_1(EF) - \text{cyc} \\
& + 2\Omega_1(AB)\Omega_1(CD)\Omega_1(EF) \\
& \vdots
\end{aligned}$$

Are these canonical forms on a different space? Yes!

[Arkani-Hamed, Henn, Trnka]

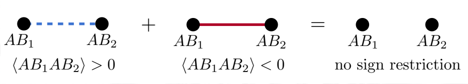
# Negative Geometries

Use the relation:

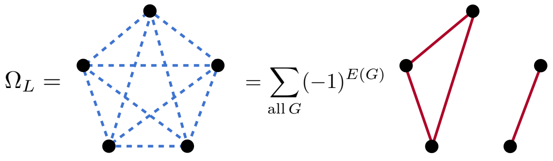


# Negative Geometries

Use the relation:

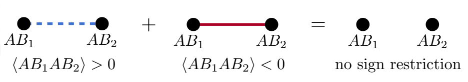



To express the amplituhedron in loop space in terms of these negative links:

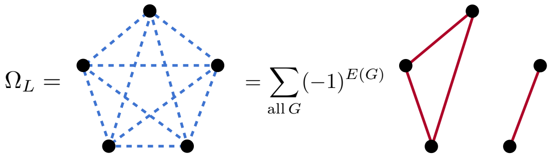


# Negative Geometries

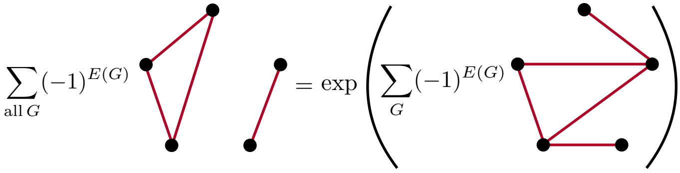
Use the relation:




To express the amplituhedron in loop space in terms of these negative links:

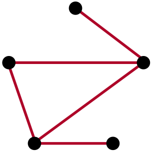



A trick familiar to us from Feynman graphs gives



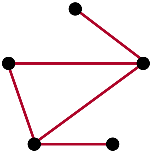
# Negative Geometries

$$\widetilde{\Omega}_L = \sum_G (-1)^{E(G)}$$

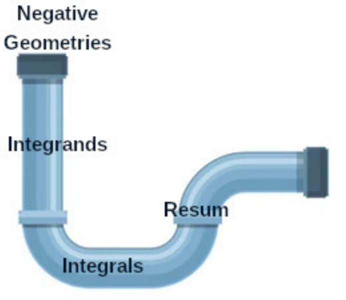


# Negative Geometries

$$\widetilde{\Omega}_L = \sum_G (-1)^{E(G)}$$



Recall our initial motivation was to get strong coupling results.




$\mathcal{F}(g, z)$

$\Gamma_{\text{cusp}}(g)$

# Ladder and Tree Resummation

Results: [Arkani-Hamed, Henn, Trnka]

$$\mathcal{F}^{\text{ladder}}(g,z) \xrightarrow{g\gg 1} 0\,, \quad \Gamma^{\text{ladder}}(g) \xrightarrow{g\gg 1} 4\sqrt{2}g - 4\frac{\log 2}{\pi} + \frac{4}{\pi}e^{-2\sqrt{2}\pi g} + \cdots$$

$$\mathcal{F}^{\text{tree}}(g,z) \xrightarrow{g\gg 1} \frac{-z}{(1+z)^2}\,, \quad \Gamma^{\text{tree}}(g) \xrightarrow{g\gg 1} \frac{8}{\pi}g + \frac{1}{\pi}\frac{1}{g} + \cdots$$

How does this compare to known results from integrability?

$r_{\text{ladder/cusp}} =$

| $g^2$ | $g^4$ | $g^6$ | $g^8$ | $g^{10}$ | $g^{12}$ | $g^{14}$ | $\cdots$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0.73 | 0.44 | 0.25 | 0.14 | 0.07 | $\cdots$ |

$r_{\text{cusp/tree}} =$

| $g^2$ | $g^4$ | $g^6$ | $g^8$ | $g^{10}$ | $g^{12}$ | $g^{14}$ | $\cdots$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0.92 | 0.83 | 0.74 | 0.63 | 0.53 | $\cdots$ |

Does 1-loop in loop space bring us closer? Our aim!

# Fixing Integrands

Denominator: Fixed by pole structure

$$\frac{1}{\prod_{i=1}^{L}\prod_{a=1}^{4}\langle AB_i aa+1\rangle} \times \frac{1}{\prod_{ij}\langle AB_i AB_j\rangle}$$

# Fixing Integrands

Denominator: Fixed by pole structure

$$\frac{1}{\prod_{i=1}^{L}\prod_{a=1}^{4}\langle AB_i aa+1\rangle} \times \frac{1}{\prod_{ij}\langle AB_i AB_j\rangle}$$

Numerator: Must vanish on unphysical cuts:

1. Cuts that violate the positivity or negativity of the links or vertices: Involve one link i.e. two vertices at a time.
2. Cuts that expose a double pole: Involve all vertices in a cycle.

Second cut is irrelevant in the tree case and as a result, the tree numerator can be built link by link.

# Examples of Cuts

Positive cuts: A cut that violates $\langle ABCD\rangle > 0$, such as

$$\begin{aligned}&\langle AB12\rangle = \langle AB34\rangle = \langle CD23\rangle = \langle CD14\rangle = 0\\ \Rightarrow & x_2 = y_1 = z_1 = w_2 = 0\\ \Rightarrow & \langle ABCD\rangle = -x_1 w_1 - y_2 z_2 < 0\end{aligned}$$

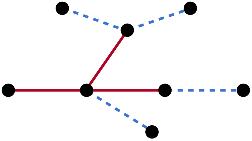
Negative cuts: A cut that violates $\langle ABCD\rangle < 0$, such as

$$\begin{aligned}&\langle AB12\rangle = \langle AB14\rangle = \langle CD23\rangle = \langle CD34\rangle = 0\\ \Rightarrow & x_1 = y_1 = z_2 = w_2 = 0\\ \Rightarrow & \langle ABCD\rangle = x_1 w_2 + y_1 z_2 > 0\end{aligned}$$

Depending on the type of link, we will require that the numerator vanishes on either type of cut.

# Tree Result

$$\Omega_G^{\text{tree}} =$$



$$= \frac{d\mu\,\mathcal{N}_G}{\prod_i D_i \cdot \prod_{\text{links}} \langle AB_i AB_j \rangle}$$

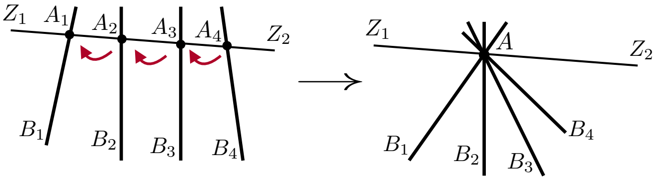
$$\mathcal{N}_G = \langle 1234 \rangle^{L+1} \times \prod_{\text{pos links}} N_{ij}^{(+)} \times \prod_{\text{neg links}} N_{ij}^{(-)}$$

$$N_{ij}^{(+)} \equiv \langle AB_i 12 \rangle \langle AB_j 34 \rangle + \text{cyc}$$

$$N_{ij}^{(-)} \equiv \langle AB_i 13 \rangle \langle AB_j 24 \rangle + \langle AB_i 24 \rangle \langle AB_j 13 \rangle$$

# Example of Cuts

Double pole cuts: Once there is a cycle, $\langle AB_i 12\rangle = 0$ and $\langle AB_i AB_{i+1}\rangle = 0$ forces $\langle AB_1 AB_L\rangle = 0$. Pictorially,




Crucially, $\mathcal{N}_{\text{tree}}$ vanishes on "positive" and "negative" cuts respectively, but not on double pole cuts.

So we need to introduce a remainder that cancels out the contribution of $\mathcal{N}_{\text{tree}}$ on double pole cuts.

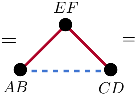
# Remainders at 1-Loop

We decompose the 1-loop numerators as

$$\prod_{\text{pos links}} N_{ij}^{(+)} \times \prod_{\text{neg links}} N_{ij}^{(-)} + R_L$$

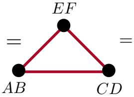
This same remainder function (up to sign) appears in different loops:

$\Omega_3^{(+--)} =$




$$= \frac{N_{12}^{(+)} N_{23}^{(-)} N_{13}^{(-)} + R_{123}}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle\langle ABCD\rangle\langle CD12\rangle\langle CD23\rangle\langle CD34\rangle \\ \langle CD14\rangle\langle CDEF\rangle\langle EF12\rangle\langle EF23\rangle\langle EF34\rangle\langle EF14\rangle\langle ABEF\rangle}$$

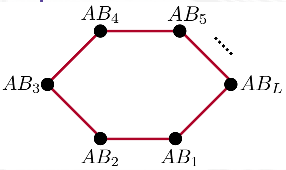
$\Omega_3^{(---)} =$




$$= \frac{N_{12}^{(-)} N_{23}^{(-)} N_{13}^{(-)} - R_{123}}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB14\rangle\langle ABCD\rangle\langle CD12\rangle\langle CD23\rangle\langle CD34\rangle \\ \langle CD14\rangle\langle CDEF\rangle\langle EF12\rangle\langle EF23\rangle\langle EF34\rangle\langle EF14\rangle\langle ABEF\rangle}$$

Thus $R$ must vanish on both "positive" and "negative" cuts $\Rightarrow$ it must be built out of a certain building blocks.

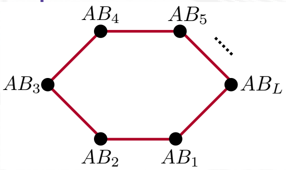
# 1-Loop Result




$$\mathcal{N}_G = \prod_{i=1}^{L} N^{(-)}_{i,i+1} + (-1)^L R_L^{1-\text{loop}}$$

$$R_L^{1-\text{loop}} = \Bigg\{(2^L-4) N_1^{(a)} N_2^{(a)} \cdots N_L^{(a)} \\ - \sum_{p\in\text{even}} \sum_i \left\{ N^{(a)}_{i_1 i_2 \ldots i_p} N^{(c)}_{i_1 i_2 \ldots i_p} N^{(a)}_{i_p+1} \cdots N^{(a)}_{i_L} \right\} \Bigg\} + (a \to b)$$

# 1-Loop Result




$$\mathcal{N}_G = \prod_{i=1}^{L} N^{(-)}_{i,i+1} + (-1)^L R_L^{1-\text{loop}}$$

$$R_L^{1-\text{loop}} = \Bigg\{(2^L-4) N_1^{(a)} N_2^{(a)} \cdots N_L^{(a)} - \sum_{p\in\text{even}} \sum_{i} \left\{ N^{(a)}_{i_1 i_2 \ldots i_p} N^{(c)}_{i_1 i_2 \ldots i_p} N^{(a)}_{i_p+1} \cdots N^{(a)}_{i_L} \right\} \Bigg\} + (a \to b)$$

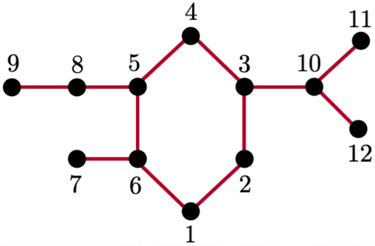
$$N^{(a)}_{12\ldots p} = \prod_{j\in\text{odd}} \langle AB_j 12\rangle \prod_{k\in\text{even}} \langle AB_k 34\rangle + \prod_{j\in\text{odd}} \langle AB_j 34\rangle \prod_{k\in\text{even}} \langle AB_k 12\rangle$$

$$N^{(b)}_{12\ldots p} = \prod_{j\in\text{odd}} \langle AB_j 23\rangle \prod_{k\in\text{even}} \langle AB_k 14\rangle + \prod_{j\in\text{odd}} \langle AB_j 14\rangle \prod_{k\in\text{even}} \langle AB_k 23\rangle$$

$$N^{(c)}_{12\ldots p} = \prod_{j\in\text{odd}} \langle AB_j 13\rangle \prod_{k\in\text{even}} \langle AB_k 24\rangle + \prod_{j\in\text{odd}} \langle AB_j 24\rangle \prod_{k\in\text{even}} \langle AB_k 13\rangle$$

# Adding Branches

Since branches do not add any new double pole conditions, it is simple to guess what the result is for a generic configuration.




$$\mathcal{N}_G = \prod_{\text{links}} N_{ij}^{(-)} + (-1)^{L'} \prod_{\text{branches}} N_{ij}^{(-)} \times R_{L'}^{1-\text{loop}}$$

# Loop Decomposition

The 1-loop answer split into two parts:

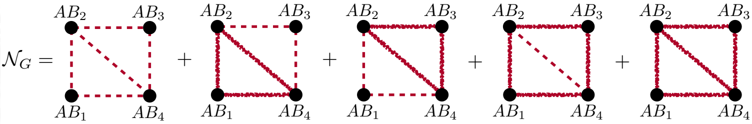
1. Part that vanishes on all "tree" cuts
1. Part that is required for "1-loop" cut

# Loop Decomposition

The 1-loop answer split into two parts:

1. Part that vanishes on all "tree" cuts
1. Part that is required for "1-loop" cut

This inspires the following general decomposition:

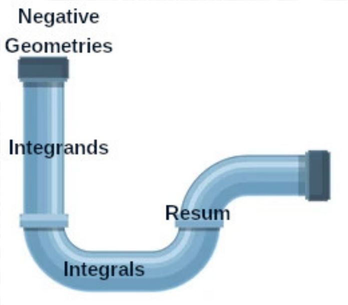



There is only one unknown piece which is

$$R^{2-\mathrm{loop}}_{1234,13} = 0 \quad \text{on any cut when } \langle AB_i AB_j \rangle \text{ has a definite sign}$$

# Moving Forward

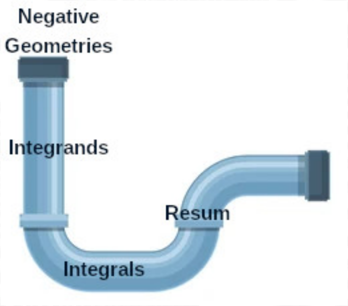
Ideally, we have the pipeline:




$\mathcal{F}(g, z)$
$\Gamma_{\text{cusp}}(g)$

# Moving Forward

Ideally, we have the pipeline:




$\mathcal{F}(g, z)$
$\Gamma_{\text{cusp}}(g)$

But for generic cycles, the integrated result is not known.

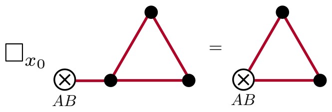
The 3-loop cycle (i.e. the triangle) is known. We can use differential equations to find triangle + branches.

# Using Differential Equations

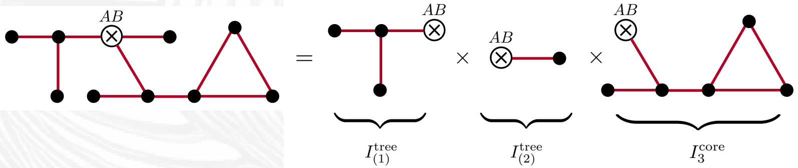
1. Action of Laplace operator

$$\Box_{x_0} \frac{1}{(x-x_0)^2} = -4\pi^2 i \delta^4(x-x_0)$$

On the integrand (rewritten in some appropriate way), this collapses a propagator:




2. Factorization occurs around the unintegrated loop:



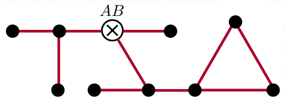
# Triangles and Friends

We can invert this Laplacian equation by assuming:

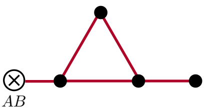
1. Result can be written in terms of HPLs
2. $\mathcal{F}(z \to -1) = 0$
3. $\mathcal{F}(\frac{1}{z}) = \mathcal{F}(z)$

The last two are consequences of the structure of the integrand.

This gives us all integrals of the type:




But not those of the type:



# Integrated Results

$$
\begin{aligned}
I_{3,1}(z) = &\, 8H_{0,0,0,0,0,0}(z) + 8H_{0,0,-1,0,0,0}(z) - 16H_{0,0,-1,-1,0,0}(z) \\
&+ 8H_{0,0,-2,0,0}(z) - 8\zeta_3(2H_{0,0,-1}(z) - H_{0,0,0}(z)) \\
&+ 4\pi^2(H_{0,0,-1,0}(z) - H_{0,0,-1,-1}(z) + H_{0,0,-2}(z)) + \frac{13\pi^4}{90}\log^2(z) \\
&+ C_{3,1}\log(z) + D_{3,1}\,,
\end{aligned}
$$

To fix the constants that, we impose the boundary conditions

$$
\begin{aligned}
C_{3,1} &= -16\zeta_{3,2} + 12\pi^2\zeta_3 - 80\zeta_5 \\
D_{3,1} &= 16\zeta_{3,3} - 16\zeta_3^2 + \frac{289\pi^6}{1890}.
\end{aligned}
$$

# Exact Results from Differential Equations

The ladder and tree results did not come from explicit resummation, but rather by solving differential equations:

$$\frac{1}{2}(z\partial_z)^2\mathcal{F}^{\text{ladder}} + g^2\mathcal{F}^{\text{ladder}} = 0$$
$$\frac{1}{2}(z\partial_z)^2\log\mathcal{F}^{\text{tree}} + g^2\mathcal{F}^{\text{tree}} = 0$$

A similar equation can be set up for the triangle.

# Exact Results from Differential Equations

The ladder and tree results did not come from explicit resummation, but rather by solving differential equations:

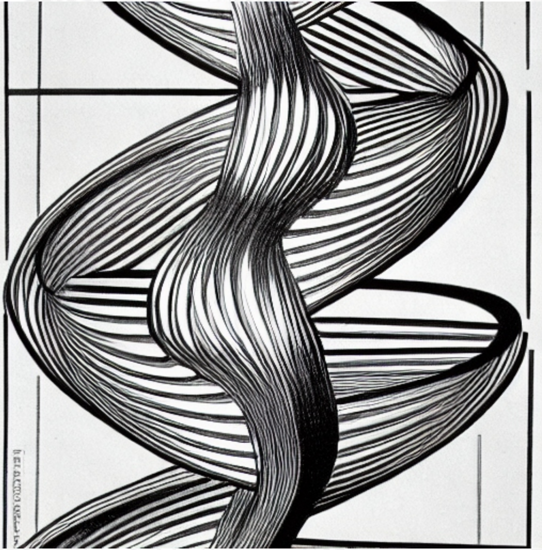
$$\frac{1}{2}(z\partial_z)^2\mathcal{F}^{\text{ladder}} + g^2\mathcal{F}^{\text{ladder}} = 0$$
$$\frac{1}{2}(z\partial_z)^2\log\mathcal{F}^{\text{tree}} + g^2\mathcal{F}^{\text{tree}} = 0$$

A similar equation can be set up for the triangle.

# Future Directions

- Can these contributions be resummed?
- Can the other 1- or higher-cycle integrands be integrated?
- What other quantities can we use similar approximations for?
- Which contributions/how many cycles capture $\mathcal{F}(g, z)$ at strong coupling?
- Do these results have nice structure at the symbol level?

$\vdots$



Thank You

# SYMply the Best Organizers