Non-planar BCFW Geometry

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Based on arxiv:22xx.xxxx
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In this talk...

We will study tree-level $\mathcal{N} = 4$ SYM via:

- BCFW recursion relations
- On-shell diagrams
- Dual Grassmannian geometry

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These methods have evinced special properties:

- Convex positive geometry
- Holomorphic forms

BCFW Recursion

Introducing a complex shift,

$$\hat{\lambda}_i = \lambda_i + z\lambda_j$$
$$\hat{\tilde{\lambda}}_j = \tilde{\lambda}_j - z\tilde{\lambda}_i$$

allows us to build amplitudes recursively.

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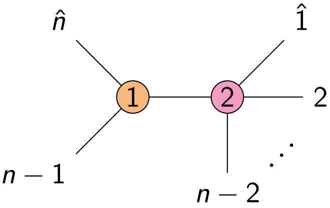
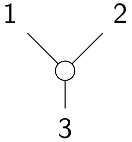
Example: $(n1)$ shift for n-point MHV amplitude has one BCFW term,



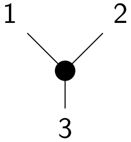
On-Shell Diagrams

$\mathcal{N} = 4$ SYM is BCFW-recursible

$\Rightarrow$ every BCFW term can be constructed by BCFW-bridging together 3-point amplitudes where external legs are on-shell,



$$= \frac{\delta^4(P)\delta^{1\times 4}(Q)}{[12][23][31]},$$



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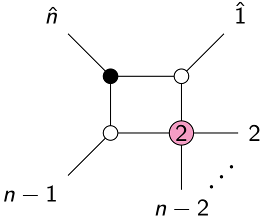
On-Shell Diagrams: MHV Example

$(n1)$ shift n-point MHV has only one BCFW term,



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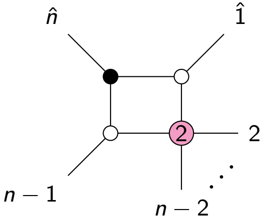
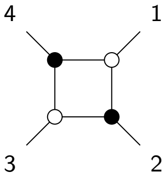
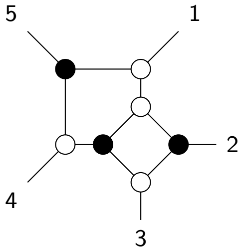


Example: At 4- and 5-point for (41) and (51) shifts,





Grassmannian

Dual formulation of $\mathcal{N} = 4$ SYM BCFW terms as auxiliary $k \times n$ matrix C,

$$\oint \frac{d^{k\times n} C}{\text{vol}\,(GL(k)) \prod \text{minors}} \delta^{2k}(C \cdot \widetilde{\lambda})\delta^{2(n-k)}(C^{\perp} \cdot \lambda)\delta^{0|4k}(C \cdot \widetilde{\eta})$$

where $C \in G_{+}(k, n)$.

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where $C\in G_{+}(k,n)$.

How is this related to on-shell diagrams?

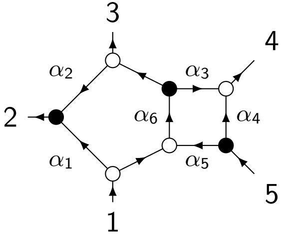
They provide a canonical parametrization α_i of C.
Resolving the δ's,

$$\frac{\delta^4(P)}{J}\delta^8(Q)\delta^{0|4(k-2)}(C^*\cdot\widetilde{\eta})\times\prod_i\frac{1}{\alpha_i}$$

Edge Variables

This canonical parametrization, can be read off the on-shell diagram for the corresponding BCFW term.

Example: (51)-shifted 5-point MHV
Decorated diagram ⇒ BCFW bridge ⇒ edge variables ⇒ C-matrix



$$C = \begin{pmatrix} 1 & \alpha_1 + \alpha_2\alpha_6 & \alpha_6 & \alpha_3\alpha_6 & 0 \\ 0 & \alpha_2\alpha_5\alpha_6 & \alpha_5\alpha_6 & \alpha_4 + \alpha_3\alpha_5\alpha_6 & 1 \end{pmatrix}$$

$$\prod_i \frac{1}{\alpha_i} \to \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$$

Multiple Descriptions Converge

In our example of $(n1)$-shifted n-point MHV,

$$\begin{aligned}\mathcal{A}_n^{\text{MHV}} =& \mathcal{A}_3^{\overline{\text{MHV}}} \times \mathcal{A}_{n-1}^{\text{MHV}} \\ =& \text{On-shell diagram} \\ =& (2n-4)\text{-dimensional cells of } G_{+}(2,n)\end{aligned}$$

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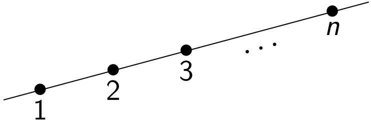
Two extra features arise:

- Positivity/convexity
- Holomorphicity

Positive Geometry

Visualize the C-matrix directly in kinemtical λ-space.

For $C \in G_{+}(2, n)$, $C = \lambda \in \mathbb{P}^1$,



$$C = \begin{pmatrix} * & * & \cdots & * \\ * & * & \cdots & * \end{pmatrix}$$

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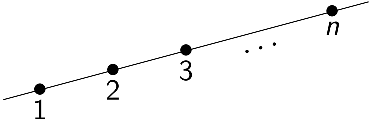


$$C = \begin{pmatrix} * & * & \cdots & * \\ * & * & \cdots & * \end{pmatrix}$$

Positivity of minors $\Rightarrow$ ordered points in $\mathbb{P}^1$.

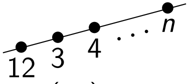
This corresponds to the Parke-Taylor amplitude,

$$\frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle \ldots \langle n1\rangle}$$

Holomorphicity

$PT(1, \cdots, n)$ only has poles when $\langle k k+1 \rangle \to 0$.

This can be seen directly from the geometry,



$(12) = 0$

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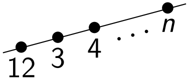
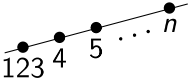


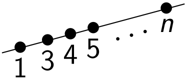
$(12) = 0$

Similarly for lower poles,



$(12) = (23) = (13) = 0$

$P_1 \sim P_2 \sim P_3$



$(2*) = 0$

$P_2 \to 0$

Now: Beyond Adjacent MHV

For adjacent-shifted NMHV and beyond:

- On-shell diagrams still correspond to cells in $G_{+}(k, n)$
- Is there a notion of holomorphicity?

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For adjacent-shifted NMHV and beyond:

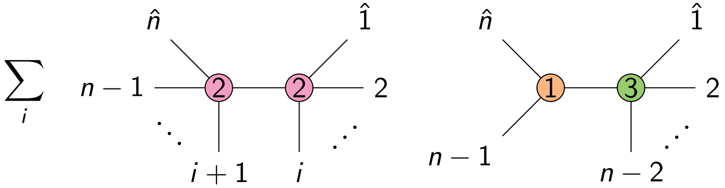
- On-shell diagrams still correspond to cells in $G_{+}(k, n)$
- Is there a notion of holomorphicity?

For non-adjacent BCFW shifts:

- Is there any residual convexity?
- What about holomorphicity?

Positive Geometry for Adjacent NMHV

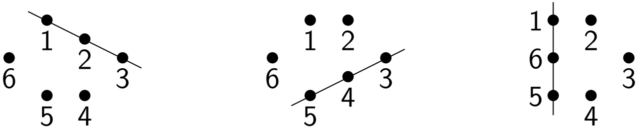
Two types of BCFW terms:



$$= \sum \text{on-shell diagrams} = \sum \text{codimension } n-5 \text{ cells of } G_{+}(3,n)$$

Example: (61) shifted NMHV_6

There are three BCFW terms that correspond to three lower-dimensional $G_{+}(3,6)$ cells,



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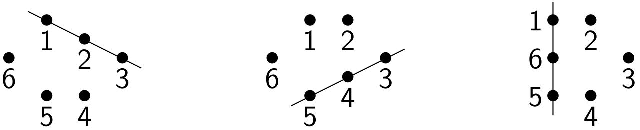


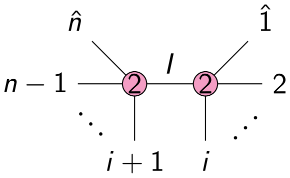
Evaluating the sum over these cells gives

$$\mathcal{A}_6^{(3)} = \mathcal{R}_1 + \mathcal{R}_3 + \mathcal{R}_5 \, .$$

where the R-invariants are

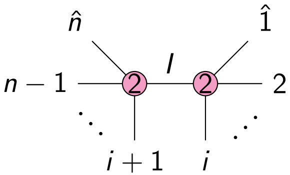
$$\mathcal{R} = \frac{\delta^4(P)\delta^8(Q)\delta^4([56]\eta_4 + [64]\eta_5 + [45]\eta_6)}{s_{123}\langle 12\rangle\langle 23\rangle[45][56]\langle 1|23|4]\langle 3|45|6]} \, ,$$

Holomorphicity for Adjacent NMHV



$$= A^{\text{tree}}_{k=2}(i{+}1, i{+}2, \ldots, n{-}1, \hat{n}, I) \times \frac{1}{(P_2 + p_n)^2} \times A^{\text{tree}}_{k=2}(I, \hat{1}, 2, \ldots, i{-}1, i)\,.$$

Holomorphicity for Adjacent NMHV



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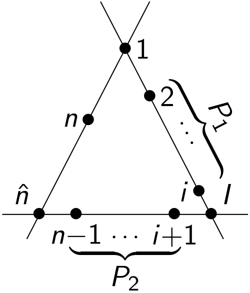
So we see that NMHV = MHV $\times$ MHV $\times$ pole which can be rewritten as MHV $\times$ MHV $\times$ MHV,

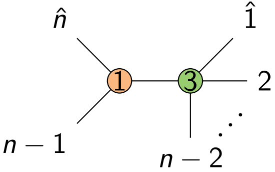


$$= PT(1, \ldots, i, I)PT(I, i{+}1, \ldots, n{-}1, \hat{n}) \times PT(\hat{n}, n, 1) \cdot \langle 1|P_1|n]^3 \times \Delta$$

where $\lambda_I = P_2|n]$

$\Rightarrow$ a holomorphic description of poles!

Holomorphicity for Adjacent NMHV



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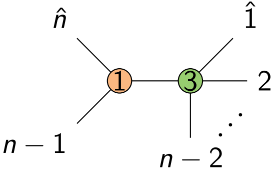
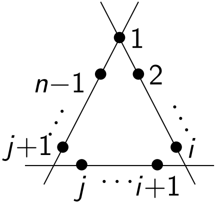
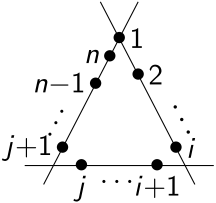


We can attach $\overline{\text{MHV}}$ to our previous expression for NMHV,

$\sum_{i,j}$

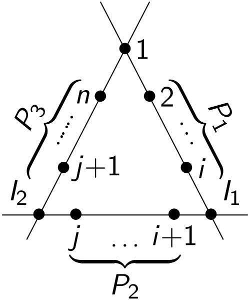


$+$ add n between $n-1$ and 1 $= \sum_{i,j}$



Holomorphicity for Adjacent NMHV

Remarkably, this extends to slightly more general configurations that do not correspond directly do BCFW geometries,



$$= PT(1,\ldots,i,l_1)PT(l_1,i{+}1,\ldots,j,l_2) \times PT(l_2,j{+}1,\ldots,n,1)\langle 1 P_1 P_3|1\rangle^3 \times \Delta$$

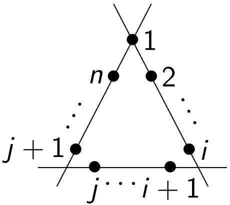
where $\lambda_{l_1} = \langle 1|P_3P_2, \lambda_{l_2} = \langle 1|P_1P_2$

Combinations of these show up in non-adjacent BCFW! Let's see how.

Aside: Reducing to Triangles

BCFW terms are a special subset of cells of $G_+(3, n)$,

$$\mathcal{A}_n^{\text{NMHV}} = \sum_{i,j}$$



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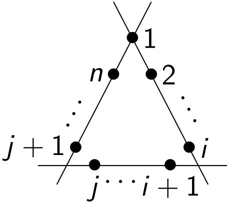
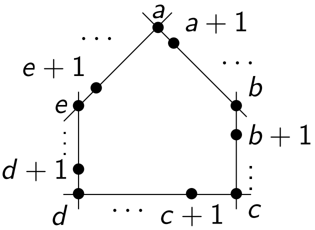
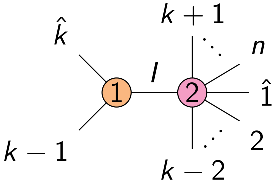


For n-point NMHV, we have $n-5$ constraints on n points in $\mathbb{P}^2$, so we are left with 5 unconstrained lines,



Non-Adjacent MHV

Already at MHV, there is more than one diagram:



Top cells are no longer convex with respect to ordering 1, 2, $\cdots$, n.

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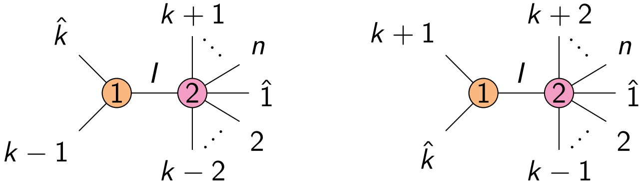
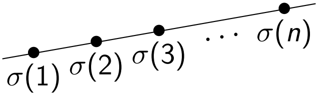


Top cells are no longer convex with respect to ordering 1, 2, $\cdots$, n.

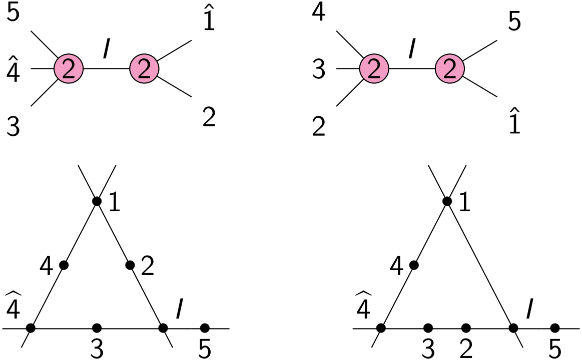
Cells are convex with respect to *some* ordering



$\Rightarrow$ Parke-Taylor factors give holomorphicity.

Non-Adjacent NMHV

Example: (14) shifted NMHV_5

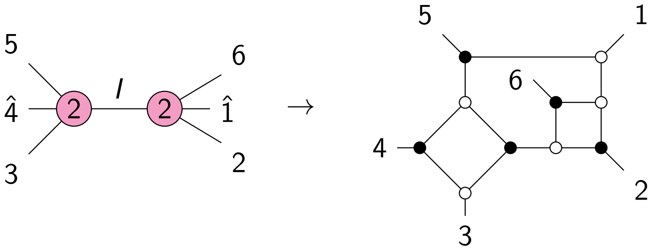


Again, convexity is lost and now it cannot be restored by changing the ordering.

Non-Planar On-Shell Diagrams

Non-adjacent BCFW terms generically correspond to non-planar on-shell diagrams.

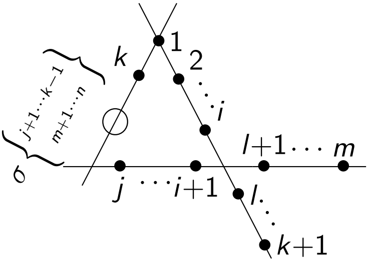
Example: (51)-shifted 6-point NMHV has a contribution from



A Lack of Positivity

For a general $(k1)$ shifted n-point amplitude,

$$A_n^{\text{NMHV}} = \sum_{\substack{i,j,l,m \\ \sigma(j+1\cdots k-1;m+1\cdots n)}}$$



Can we restore any kind of convexity?

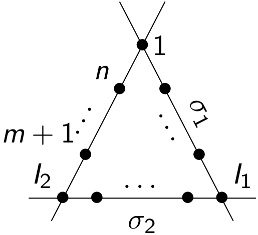
Positivity in Non-Adjacent NMHV

Kleiss-Kuijf relations read

$$\mathcal{A}_n[1,\alpha,n,\beta] = (-1)^{|\beta|} \sum_{\sigma\in\alpha \sqcup\!\sqcup\, \beta} \mathcal{A}_n[1,\sigma,n]$$

KK on each of the MHV lines allows us to rewrite each of the above non-planar non-convex geometries as a sum of planar convex ones!

$$\sum_{\sigma_1,\sigma_2}$$



where

$$\sigma_1 = \{2,\ldots,i\} \sqcup\!\sqcup \{i{+}1,\ldots,j\}^T, \qquad \sigma_2 = \{j{+}1,\ldots,k\} \sqcup\!\sqcup \{k{+}1,\ldots,m\}^T$$

Non-Adjacent BCFW and Holomorphicity

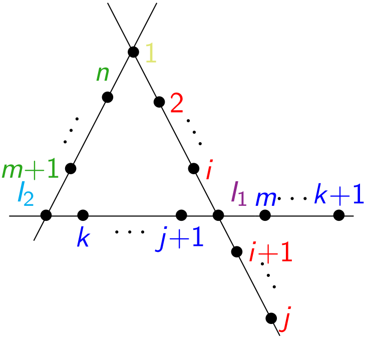
Two options:

- Non-planar geometry $= \sum$ convex configurations
  $= \sum$ products of Parke-Taylor factors for particular external orderings

Non-Adjacent BCFW and Holomorphicity

Two options:

- Non-planar geometry $= \sum$ convex configurations $= \sum$ products of Parke-Taylor factors for particular external orderings
- Non-planar $\mathcal{R}$-invariants that are defined directly on the non-convex geometry



$$
\begin{aligned}
=&PT(1,2,\ldots,i,l_1,i{+}1,\ldots,j)\\
&\times PT(l_1,j{+}1,\ldots,k,l_2,k{+}1,\ldots,m)\\
&\times PT(l_2,m{+}1,\ldots,n,1)\\
&\times \langle 1P_1P_3|1\rangle^3\Delta
\end{aligned}
$$

Summary

- Before: Adjacent BCFW-shifted MHV amplitudes enjoyed an associated positive geometry and holomorphic forms on $G_+(2, n)$

Summary

- Before: Adjacent BCFW-shifted MHV amplitudes enjoyed an associated positive geometry and holomorphic forms on $G_+(2, n)$
- After: NMHV and beyond also have associated holomorphic forms on $G_+(k, n)$ cells
- Non-adjacent BCFW gives rise to a subset of non-planar on-shell diagrams for N^kMHV
- These non-positive BCFW geometries can be expressed as a sum of positive geometries with different external particle orderings
- Associated non-planar holomorphic forms and $\mathcal{R}$-invariants can be constructed

Outlook

- Do non-planar BCFW geometries exhibit any residual dual conformal symmetry?
- What can we say about non-BCFW non-planar on-shell diagrams?
- Can we use a similar geometric construction to enumerate equivalence classes of top cells?

$\vdots$

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Thank you for listening!