One-Cycle Negative Geometries

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Based on arXiv:2604.22683 with M Skowronek, M Spradlin, A Volovich, H-C Weng
and arXiv:2605.xxxx with L Dixon, U Oktem, J Trnka, S-Q Zhang, Y Xu

$\mathcal{N} = 4$ SYM

- Color-ordered amplitudes have fewer singularities.
- Tree amplitudes are constructible recursively.

  [Britto, Cachazo, Feng, Witten]
- It has dual conformal symmetry.

  [Drummond, Korchemsky, Sokatchev]
- It has a holographic dual which one can check answers against or make predictions for.

  [Maldacena]
- There exists a polytope description of its Feynman integrands (amplituhedron).

  [Arkani-Hamed, Trnka]
- Some integrals lead to a space of functions with a well-defined symbol.

  [Goncharov, Spradlin, Vergu, Volovich]
- Symbol bootstrap via properties such as cluster adjacency.

  [Drummond, Foster, Gürdoǵan]

$\vdots$

IR Singularities

At L-loop, the amplitude has divergences, $\mathcal{M}^{(L)} \sim \frac{1}{\epsilon^{2L}}$.

Divergences were shown to have special structure $\Rightarrow$ BDS Ansatz:

non-iterative terms

$E_n^{(l)}(0) = 0$

$$\mathcal{M}_4 = \exp\left[\sum_{l=1}^{\infty} g^l \left(f^{(l)}(\epsilon)\mathcal{M}_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)\right)\right]$$

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$f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$

constant

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constant

cusp anomalous dimension

IR Finite Quantities

BDS ansatz suggests we should look at

$$\log \mathcal{M} = \frac{\Gamma_{\mathsf{cusp}} \mathcal{M}^{(1)}_{(-2)}}{\epsilon^2} + \mathcal{O}(\epsilon)$$

The divergence is milder. But is there an IR finite quantity we can compute instead?

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Consider the integrand of $\log \mathcal{M}_4^{(L)} = \int \tilde{\Omega}_L$.

Since each integration gives $\frac{1}{\epsilon^2}$, doing all but one integral gives an IR finite quantity that can only depend on

$$z = \frac{\langle AB12 \rangle \langle AB34 \rangle}{\langle AB14 \rangle \langle AB23 \rangle}$$

due to dual conformal invariance of the integrand.

A Third Loop

We denote:

$$\mathcal{F}_L(z) = \int_{\text{all but one}} \tilde{\Omega}_L \quad \Rightarrow \quad \mathcal{F}(g,z) = \sum_{L=1}^{\infty} \mathcal{F}_L(z) g^{2L}$$

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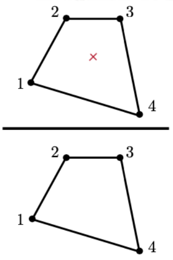
Wilson loop $\leftrightarrow$ amplitude duality tells us that this quantity is

$\mathcal{F}(g,z) =$



[Drummond, Korchemsky, Sokatchev, Alday, Buchbinder, Tsytelin, Engelund, Roiban, Henn, Sikorowski, Korchemsky, Mistlberger, Chicherin...]

This integrand $\mathcal{F}(g,z)$ can be used to obtain Γ_{cusp}.

Known Asymptotic Behaviour

Results from integrability, but no strong coupling results from amplitudes:

[Beisert, Eden, Staudacher]

$$\mathcal{F}(g,z) \to \begin{cases} -g^2 + g^4(\log^2 z + \pi^2) + \cdots & g \ll 1 \\ g\frac{z}{(z-1)^3}\left(2(1-z) + (z+1)\log z\right) + \cdots & g \gg 1 \end{cases}$$

As we saw this is related to a particular ratio of Wilson loops.

[Alday, Henn, Sirkowski] [Henn, Korchemsky, Mistlberger]

$$\Gamma(g) \to \begin{cases} 4g^2 - 8\zeta_2 g^4 + \cdots & g \ll 1 \\ 2g - \frac{3\log 2}{2\pi} + \cdots & g \gg 1 \end{cases}$$

Perturbative regime results can be reproduced from amplitudes.

[Bern, Czakon, Dixon, Kosower, Smirnov]

Strong coupling regime results can be reproduced from minimal surface anchored to the boundary at a null polygon.

[Benna, Benvenuti, Klebanov, Scardicchio] [Basso,Korchemsky, Kotanski]

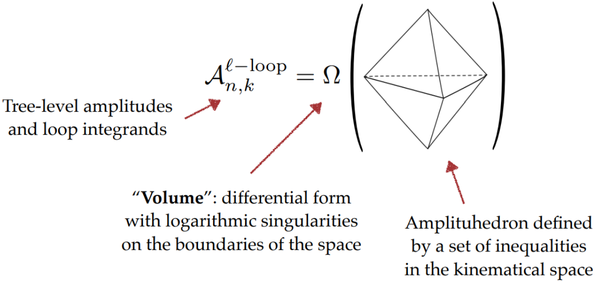
[Alday, Buchbinder, Tsytelin]

This Talk

1. Can we get all-loop results for $\mathcal{F}(z)$ and Γ_{cusp}?
2. Can we capture $g \gg 1$ behaviour from the all-loop integrand given via the amplituhedron?
3. Can we better understand the function space of $\mathcal{F}(z)$ and the number space of Γ_{cusp}?

Method: Amplituhedron

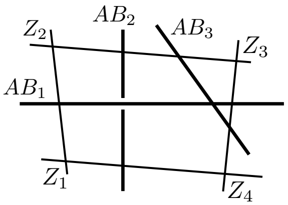
The amplituhedron is a polytope in twistor space whose canonical form ("volume") gives tree amplitudes and loop integrands in $\mathcal{N} = 4$ SYM.



[Arkani-Hamed, Trnka]

Four-Point Amplituhedron*

At L-loop, introduce L lines in momentum twistor space,



$Z_i \to$ External momentum twistors
$AB_i \to$ Loop lines

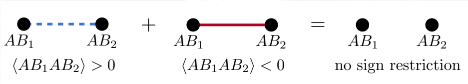
Z_i satisfy $\langle 1234 \rangle > 0$, while AB_i satisfy the following conditions:

$$\langle AB_i a a{+}1 \rangle > 0 \quad \langle AB_i 13 \rangle < 0 \quad \langle AB_i 24 \rangle < 0 \quad \langle AB_i AB_j \rangle > 0$$

We are looking for a canonical form on the space carved out by these inequalities i.e. a form that has logarithmic singularities on all the boundaries and at no other location.

Negative Geometries

Use the relation:



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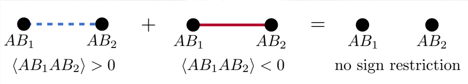
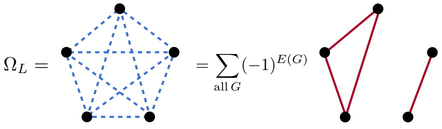


To express the amplituhedron in loop space in terms of these negative links:



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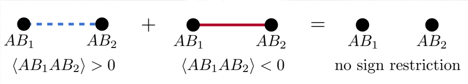

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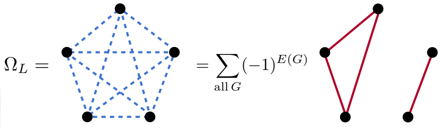
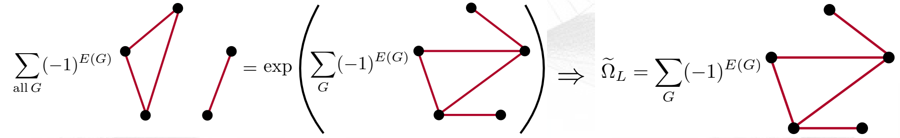


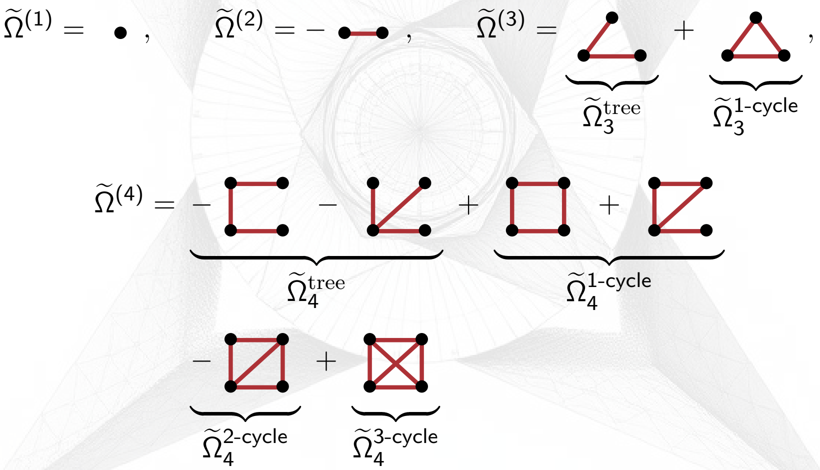
A trick familiar to us from Feynman graphs gives



Negative Geometries

Clearly the negative geometry expansion organizes itself by

- No. of loops $L =$ no. of nodes (controlled by 't Hooft coupling g)
- No. of cycles $=$ no. of loops of loops (controlled by ?)



Motivation: Ladder and Tree Resummation

[Arkani-Hamed, Henn, Trnka]

Results:

$$\mathcal{F}^{\text{ladder}}(g,z) \xrightarrow{g\gg 1} 0\,, \quad \Gamma^{\text{ladder}}(g) \xrightarrow{g\gg 1} 4\sqrt{2}g - 4\frac{\log 2}{\pi} + \frac{4}{\pi}e^{-2\sqrt{2}\pi g} + \cdots$$

$$\mathcal{F}^{\text{tree}}(g,z) \xrightarrow{g\gg 1} \frac{-z}{(1+z)^2}\,, \quad \Gamma^{\text{tree}}(g) \xrightarrow{g\gg 1} \frac{8}{\pi}g + \frac{1}{\pi}\frac{1}{g} + \cdots$$

$$\mathcal{F}^{\text{known}}(g,z) \xrightarrow{g\gg 1} \mathcal{O}(g)\,, \quad \Gamma^{\text{known}}(g) \xrightarrow{g\gg 1} 2g - \frac{3\log 2}{2\pi} + \cdots$$

How does this compare to known perturbative results from integrability?

$r_{\text{ladder/cusp}} =$

g^2	g^4	g^6	g^8	g^{10}	g^{12}	g^{14}	$\cdots$
1	1	0.73	0.44	0.25	0.14	0.07	$\cdots$

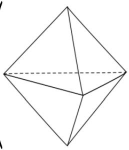
$r_{\text{cusp/tree}} =$

g^2	g^4	g^6	g^8	g^{10}	g^{12}	g^{14}	$\cdots$
1	1	0.92	0.83	0.74	0.63	0.53	$\cdots$

Does 1-loop in loop space bring us closer? Our aim!

Fixing Integrands

$$\mathcal{A}^{\ell-\text{loop}}_{n,k} = \Omega\left(\quad \right)$$



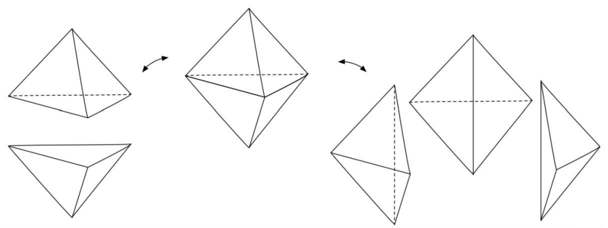
Tree-level amplitudes and loop integrands

"**Volume**": differential form with logarithmic singularities on the boundaries of the space

Amplituhedron defined by a set of inequalities in the kinematical space

[Arkani-Hamed, Trnka]

Fixing Integrands



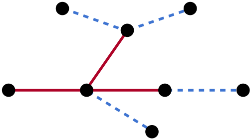
[Arkani-Hamed, Trnka]

In practice, calculating this volume form involves triangulation which is non-trivial. One can instead take an algebraic approach of requiring:

1. Cuts that violate the positivity or negativity of the links or vertices: Involve at most one link i.e. two vertices at a time.
2. Cuts that expose a double pole: Involve all vertices in a cycle.

Tree Result

$$\Omega_G^{\text{tree}} =$$



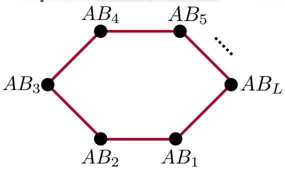
$$= \frac{d\mu\, \mathcal{N}_G}{\prod_i D_i \cdot \prod_{\text{links}} \langle AB_i AB_j \rangle}$$

$$\mathcal{N}_G = \langle 1234 \rangle^{L+1} \times \prod_{\text{pos links}} N_{ij}^{(+)} \times \prod_{\text{neg links}} N_{ij}^{(-)}$$

$$N_{ij}^{(+)} \equiv \langle AB_i 12 \rangle \langle AB_j 34 \rangle + \text{cyc}$$

$$N_{ij}^{(-)} \equiv \langle AB_i 13 \rangle \langle AB_j 24 \rangle + \langle AB_i 24 \rangle \langle AB_j 13 \rangle$$

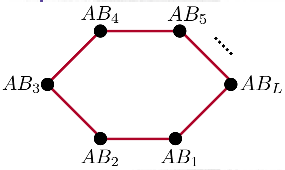
1-Loop Result



$$\mathcal{N}_G = \prod_{i=1}^{L} N^{(-)}_{i,i+1} + (-1)^L R^{1-\text{loop}}_L$$

$$R^{1-\text{loop}}_L = \Bigg\{(2^L-4) N^{(a)}_1 N^{(a)}_2 \cdots N^{(a)}_L - \sum_{p\in\text{even}} \sum_i \left\{ N^{(a)}_{i_1 i_2 \ldots i_p} N^{(c)}_{i_1 i_2 \ldots i_p} N^{(a)}_{i_p+1} \cdots N^{(a)}_{i_L} \right\} \Bigg\} + (a \to b)$$

1-Loop Result



$$\mathcal{N}_G = \prod_{i=1}^{L} N^{(-)}_{i,i+1} + (-1)^L R_L^{1-\text{loop}}$$

$$R_L^{1-\text{loop}} = \Bigg\{ (2^L - 4) N_1^{(a)} N_2^{(a)} \cdots N_L^{(a)} - \sum_{p\in\text{even}} \sum_{i} \left\{ N^{(a)}_{i_1 i_2 \ldots i_p} N^{(c)}_{i_1 i_2 \ldots i_p} N^{(a)}_{i_p+1} \cdots N^{(a)}_{i_L} \right\} \Bigg\} + (a \to b)$$

$$N^{(a)}_{12\ldots p} = \prod_{j\in\text{odd}} \langle AB_j 12\rangle \prod_{k\in\text{even}} \langle AB_k 34\rangle + \prod_{j\in\text{odd}} \langle AB_j 34\rangle \prod_{k\in\text{even}} \langle AB_k 12\rangle$$

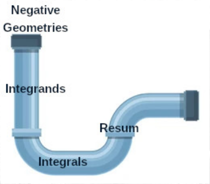
$$N^{(b)}_{12\ldots p} = \prod_{j\in\text{odd}} \langle AB_j 23\rangle \prod_{k\in\text{even}} \langle AB_k 14\rangle + \prod_{j\in\text{odd}} \langle AB_j 14\rangle \prod_{k\in\text{even}} \langle AB_k 23\rangle$$

$$N^{(c)}_{12\ldots p} = \prod_{j\in\text{odd}} \langle AB_j 13\rangle \prod_{k\in\text{even}} \langle AB_k 24\rangle + \prod_{j\in\text{odd}} \langle AB_j 24\rangle \prod_{k\in\text{even}} \langle AB_k 13\rangle$$

[Brown, Oktem, SP, Trnka]

Moving Forward

Ideally, we have the pipeline: But for 1-cycle L-loop, the integrated result is not known.



$\mathcal{F}(g, z)$
$\Gamma_{\text{cusp}}(g)$

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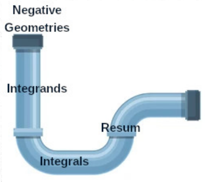
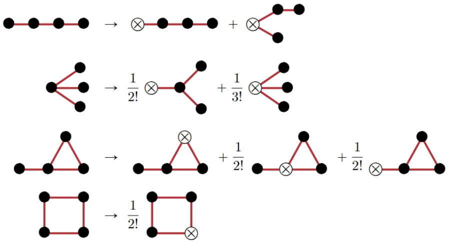
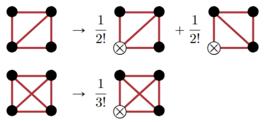


$\mathcal{F}(g, z)$
$\Gamma_{\text{cusp}}(g)$

New full 4-loop:

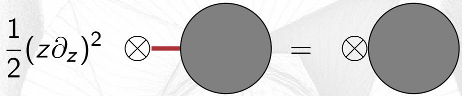




[Dixon, Oktem, SP, Trnka, Zhang, Xu] (to appear)

Triangles and Friends

The 3-loop and 4-loop (new!) cycle (i.e. the triangle and square negative geometries) are known. We can use differential equations to find triangle/box + branches.



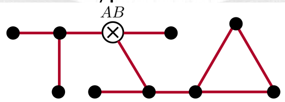
To solve the differential equations, we use

- The boundary condition

$$\mathcal{F}(z \to -1) = 0$$

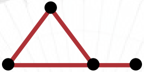
- The fact that the 3-loop 1-cycle result only contains harmonic polylogarithms (HPLs).

This gives us all integrals of the type:



[Brown, Oktem, SP, Trnka]

Integrated Results: Single Branch

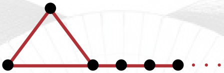


$$
\begin{aligned}
\mathcal{F}_{\triangle,1}(z) = {} & 8H_{0,0,0,0,0,0}(z) + 8H_{0,0,-1,0,0,0}(z) - 16H_{0,0,-1,-1,0,0}(z) \\
& + 8H_{0,0,-2,0,0}(z) - 8\zeta_3(2H_{0,0,-1}(z) - H_{0,0,0}(z)) \\
& + 4\pi^2(H_{0,0,-1,0}(z) - H_{0,0,-1,-1}(z) + H_{0,0,-2}(z)) + 13\zeta_4 \log^2(z) \\
& + C_{3,1}\log(z) + D_{3,1}\,,
\end{aligned}
$$

To fix the constants that, we impose the boundary conditions

$$
\begin{aligned}
C_{3,1} &= 8\zeta_5 + 24\zeta_2\zeta_3 \\
D_{3,1} &= -8\zeta_3^2 + 155\zeta_6.
\end{aligned}
$$

Integrated Results: n-Branch Triangle



$$\mathcal{F}_{\triangle,n}(z) = (-1)^n \Bigg[8H_{\vec{0}_{2n},0,0,0,0}(z) + 8H_{\vec{0}_{2n},-1,0,0,0}(z) - 16H_{\vec{0}_{2n},-1,-1,0,0}(z) + 8H_{\vec{0}_{2n},0,-1,0,0}(z)$$
$$- 16\zeta_3\, H_{\vec{0}_{2n},-1}(z) + 4\pi^2 \left(H_{\vec{0}_{2n},-1,0}(z) - 2H_{\vec{0}_{2n},-1,-1}(z) + H_{\vec{0}_{2n},-2}(z) \right)$$
$$+ 26\zeta_4 \frac{1}{(2n)!} \log^{2n}(z) + 8\zeta_3 \frac{1}{(2n+1)!} \log^{2n+1}(z)$$
$$+ \sum_{k=1}^{n} (-1)^k \left(\frac{C_k}{[2(n-k)+1]!} \log z^{2(n-k)+1} + \frac{D_k}{[2(n-k)]!} \log z^{2(n-k)} \right) \Bigg].$$

where the constants are fixed recursively using the boundary conditions.

Note: Only harmonic polylogarithms appear here i.e. the symbol letters are z, $z+1$.

[Dixon, Oktem, SP, Trnka, Zhang, Xu] (to appear)

Extracting Γ_{cusp}

We have another integral to perform,

$$g\frac{\partial}{\partial g}\Gamma_{\text{cusp}} = \oint F(z)dz$$

Integrating gives,

	g^2	g^4	g^6	g^8	g^{10}	g^{12}	g^{14}
$\Gamma_{\text{tree}}/\Gamma_{\text{true}}$	1	1	1.091	1.198	1.353	1.579	1.893
$\Gamma_{\triangle,n}/\Gamma_{\text{true}}$	0	0	-0.091	-0.053	-0.059	-0.074	-0.097
$\Gamma_{\text{tree},\triangle,n}/\Gamma_{\text{true}}$	1	1	1	1.145	1.294	1.505	1.796

[Dixon, Oktem, SP, Trnka, Zhang, Xu] (to appear)

Γ_{cusp} for n-branch Triangle

$$\Gamma_{\mathrm{cusp}}(\mathcal{F}_{3,n}(z)) = -\frac{4g^{2(n+3)}}{(n+3)}\Big(\alpha_n + 32\,(-1)^{n+1}\sum_{k=1}^{n} k\, f_{2(n-k)+3,2k+1}$$
$$+ (-1)^{n+1}\sum_{j=0}^{n}\xi_{j+1}\pi^{2j}\sum_{k=1}^{n-j} k(2k-1)\, f_{2(n-j-k)+3,2k+1}\Big)\,,$$

where α_n is purely made of even zeta values, i.e. powers of π.

It is clear that this result has uniform transcendentality of $2n+4$ i.e. $2L$.

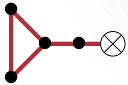
It also has MZV's which are presumed to cancel in the full sum. 5-loop is the first order at which we can test such a cancellation.

Similar formula available for box + n-branch.

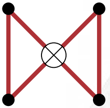
[Dixon, Oktem, SP, Trnka, Zhang, Xu] (to appear)

Some 5-loop contributions

Consider the following two 5-loop contributions explicitly:

$\Gamma_{\text{cusp}}\Big[$  $\Big]$

$$= \frac{4g^{10}}{5}\left(-\frac{448}{5}\zeta_{5,3} - \frac{1}{3}32\pi^2\zeta_3^2 - 640\zeta_5\zeta_3 + \frac{9944\pi^8}{70875}\right)$$

$\Gamma_{\text{cusp}}\Big[$  $\Big]$

$$= \frac{4g^{10}}{5}\left(\frac{406888}{45}\zeta_8 + 384\zeta_2\zeta_3^2 - 7680\zeta_3\zeta_5 - \frac{13056}{5}\zeta_{5,3}\right)$$

Both of these have MZV's that will have to cancel against the other contributions.

Resumming Γ_{cusp}

Since we know the resummation of the ladder, we can use it to calculate the resummation of an infinite series of contributions to the cusp anomalous dimension:

$$\oint F_{\triangle}(z)F_{\text{ladder}}(z) = g\frac{\partial}{\partial g}\Gamma_{\text{cusp}}\left[\quad\right]$$

$$\Gamma_{\text{cusp}} \overset{g\gg 1}{\sim} -\frac{88}{225}\pi^3\sqrt{2}g^5 + \mathcal{O}(g^4)$$

Does this mean that the expansion will not work at strong coupling?

Maybe, but this is not all 1-cycle contributions.

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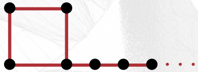
The contribution from the box is

$$\oint F_{\square}(z) F_{\text{ladder}}(z) = g \frac{\partial}{\partial g} \Gamma_{\text{cusp}} \left[\quad \right]$$



$$\overset{g \gg 1}{\sim} \frac{3152\sqrt{2}\pi^5}{6615} g^7 + \mathcal{O}(g^4)$$

Suggestive that 1-cylce contributions could resum?

Higher-loop 1-cycle

For both

- MZV cancellation
- Resummation of gamma cusp contributions

we find that it would be useful to go to higher loop order within the class of 1-cycle geometries.

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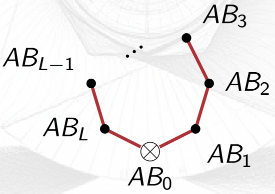
Where can we start thinking about other 1-cycle contributions? n-gons



What can we attempt to say about them? Singularities i.e. are they HPL?

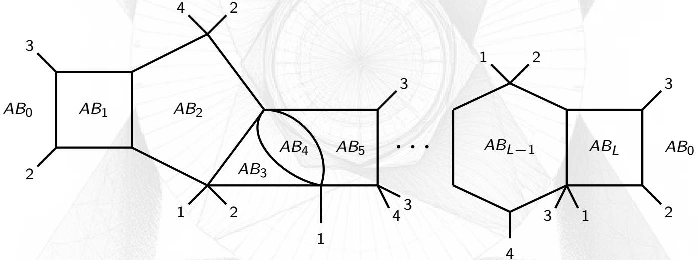
Geometric Landau Analysis for 1-Cycle Geometries

Landau equations:

Cut conditions: $\alpha_e q_e^2 = 0$ for every edge

Pinch conditions: $\sum_e \alpha_e q_e^\mu = 0$ for each loop

Loci can be represented by a Landau diagram:

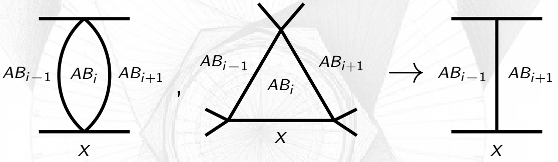


[Dennen, Prlina, Spradlin, Stanojevic, Volovich]

[Chicherin, Henn, Mazzucchelli, Trnka, Yang, Zhang]

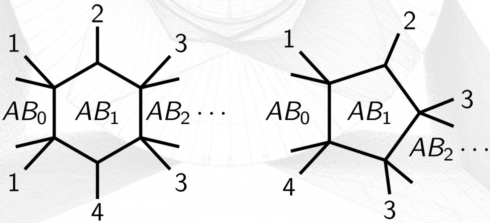
Proof of $z = 0, -1, \infty$ as singularities

- Bubbles and triangles can be eliminated



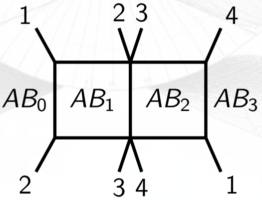
Proof of $z = 0, -1, \infty$ as singularities

- Bubbles and triangles can be eliminated.
- Hexagons have no solution.
- Pentagons can be eliminated by considering the cases of what the pentagon is attached to : four-mass box (no solution), three-mass box (singularity at $z = -1$) or a pentagon (repeat argument recursively).
- Three pentagons in a row: no solution.



Proof of $z = 0, -1, \infty$ as singularities

- Bubbles and triangles can be eliminated.
- Hexagons have no solution.
- Pentagons only lead to $z = -1$.
- First loop is on a 4-mass box: has a solution for $z = 0, \infty$.
- First loop is on a 3-mass box: Second loop can only be an "opposite" 3-mass box, or else $z = 0$ is a solution, and the two 3-mass boxes together only have a solution for $z = 0, -1, \infty$.



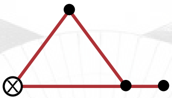
Proof of $z = 0, -1, \infty$ as singularities

- Bubbles and triangles can be eliminated.
- Hexagons have no solution.
- Pentagons only lead to $z = -1$.
- First loop is on a 4-mass box: $z = 0, \infty$.
- First loop is on a 3-mass box: $z = 0, -1, \infty$.

The only Landau singularities are at $z = 0, -1, \infty$!

[SP, Skowronek, Spradlin, Volovich, Weng]

Caveat: Other letters



$\mathrm{Symbol}(I_{3,2}) =$

$$12\,\big(8\,\mathrm{SB}(z,z,z,z,b,b) + 2\,\mathrm{SB}(z,z,z,b,z,b) + 2\,\mathrm{SB}(z,z,z,b,b,z)$$
$$+\,28\,\mathrm{SB}(z,z,z,z,z,z) + 8\,\mathrm{SB}(z,z,z,z,z,1{+}z) + 16\,\mathrm{SB}(z,z,z,z,1{+}z,z)$$
$$-\,8\,\mathrm{SB}(z,z,z,z,1{+}z,1{+}z) + 18\,\mathrm{SB}(z,z,z,1{+}z,z,z) - 2\,\mathrm{SB}(z,z,z,1{+}z,z,1{+}z)$$
$$-\,2\,\mathrm{SB}(z,z,z,1{+}z,1{+}z,z) + 30\,\mathrm{SB}(z,z,1{+}z,z,z,z) - 6\,\mathrm{SB}(z,z,1{+}z,z,z,1{+}z)$$
$$-\,22\,\mathrm{SB}(z,z,1{+}z,z,1{+}z,z) - 32\,\mathrm{SB}(z,z,1{+}z,1{+}z,z,z)\big)\,,$$

where

$$b = \frac{1+i\sqrt{z}}{1-i\sqrt{z}}\,.$$

The additional letter b shows up in individual negative geometries (starting from 4th entry), but drops out in the $F^{(3)}$ after sum.

Features of the Result

- Γ_{cusp} for tree graphs at L-loop has transcendentality $2L$ with single zeta values, but no MZVs.
- As the number of cycles increases, the depth of the MZVs also increase e.g. 1-cycle has MZVs of depth up to 2.
- Known integrated 1-cycle results are made of HPL's rather than GPL's (spurious letters that appear at higher-cycle cancel). We can now prove the the only letters are z, $1+z$ and b.
- Higher the weight of the seed HPL, higher the expected g power of the resummed contribution at strong coupling.
- Geometric series contribute at sub-leading order in strong coupling.

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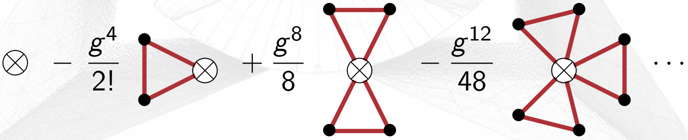
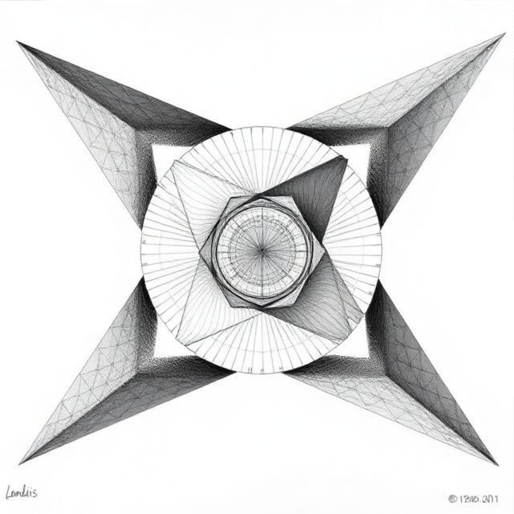


Future Directions

- Can the other 1- or higher-cycle integrands be integrated?
- Can all 1-cycle contributions be resummed?
- If yes, will that give us a "better approximation" to the cusp anomalous dimension at strong coupling?
- Can we use cuts of the integrand to understand which combinations of graph symbols do not depend on b?
- Can geometric Landau analysis be used for all 1-cycle geometries to show that the only singularities are at $z = 0, -1, \infty$?

$\vdots$



Děkuju!